Quasilinear Model of Magnetic Field Penetration of Dynamic Ergodic Divertor Fields*

Martin F. Heyn[1], Christian G. Eherer[1], Winfried Kernbichler[1], and Sergei V. Kasilov[2]

[1] *Institut für Theoretische Physik, Technische Universität Graz, Petersgasse 16, A–8010 Graz, Austria*
[2] *Institute of Plasma Physics, National Science Center "Kharkov Institute of Physics and Technology", Ul. Akademicheskaya 1, 61108 Kharkov, Ukraine*

Introduction

The DED coil system of TEXTOR will generate electromagnetic waves which penetrate into the plasma volume. These waves induce currents within the plasma, in particular near the resonant surface $q = 3$ located at the edge. The coupling between the DED fields and the plasma is of high interest in view of ergodization of the magnetic field (transport), shielding of core plasma, coupling of momentum to the plasma fluid (rotating plasma) [1]. It is assumed that the physical processes can be studied within the 2-fluid description of an electron-ion plasma in the following way; in a first step the small amplitude DED perturbations are calculated in the linear approximation; in a second step, the "slow" evolution of the background is calculated in the quasilinear approximation. The source terms of the quasilinear equations are obtained from the solution of the linear wave problem.

Basic Equations

The tokamak is described as a periodic cylinder and Fourier expansion is done with respect to time and angles ($\frac{\partial}{\partial t} \to i\omega$, $(\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial z}) \to i\mathbf{k}$). As a result, a set of ordinary differential equations in the radial variable r for the respective Fourier amplitudes is obtained from Maxwell equations, fluid momentum conservation, mass conservation and energy conservation. In the following, prime means radial derivative, subscript $\|$ and $\perp$ projections with respect to $\mathbf{h} = \mathbf{B}_0/B_0$ and $\mathbf{h}\times\mathbf{e}_r$, $\eta_\alpha^{0,3,4}$ are the viscosity coefficients and the $W_{0,3,4}^{ij}$ are related to the rate of strain tensor [2],

$$
\begin{aligned}
B_r &= -N_\| E_\perp + N_\perp E_\|, \qquad E_r = N_\| B_\perp - N_\perp B_\| + \frac{4\pi}{i\omega} j_r, \\
\frac{c}{i\omega} E'_\| &= -B_\perp + N_\| E_r + \frac{c}{i\omega}\left(\mathbf{e}_\perp \cdot \mathbf{h}' - \frac{h_\vartheta h_z}{r}\right) E_\perp - \frac{c}{i\omega}\frac{h_\vartheta^2}{r} E_\|, \\
\frac{c}{i\omega} B'_\| &= E_\perp + N_\| B_r + \frac{c}{i\omega}\left(\mathbf{e}_\perp \cdot \mathbf{h}' - \frac{h_\vartheta h_z}{r}\right) B_\perp - \frac{c}{i\omega}\frac{h_\vartheta^2}{r} B_\| - \frac{4\pi}{i\omega} j_\perp, \\
\frac{c}{i\omega} E'_\perp &= B_\| + N_\perp E_r - \frac{c}{i\omega}\left(\mathbf{e}_\perp \cdot \mathbf{h}' + \frac{h_\vartheta h_z}{r}\right) E_\| - \frac{c}{i\omega}\frac{h_z^2}{r} E_\perp \\
\frac{c}{i\omega} B'_\perp &= -E_\| + N_\perp B_r - \frac{c}{i\omega}\left(\mathbf{e}_\perp \cdot \mathbf{h}' + \frac{h_\vartheta h_z}{r}\right) B_\| - \frac{c}{i\omega}\frac{h_z^2}{r} B_\perp + \frac{4\pi}{i\omega} j_\|.
\end{aligned}
$$

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$$
\begin{aligned}
&(-i\omega + i\mathbf{k}\cdot\mathbf{v}_{\alpha0})\, v_{\alpha r} - \frac{2}{r} v_{0\vartheta}\left(h_z v_{\alpha\perp} + h_\vartheta v_{\alpha\|}\right) - \frac{n_\alpha}{n_{\alpha0}^2}\left(n_{\alpha0}\frac{T_{\alpha0}}{m_\alpha}\right)' \\
&+ \frac{1}{n_{\alpha0}}\left(n_{\alpha0}\frac{T_\alpha}{m_\alpha} + n_\alpha\frac{T_{\alpha0}}{m_\alpha}\right)' - \frac{1}{n_\alpha m_\alpha}F_{\alpha r} = \frac{e_\alpha}{m_\alpha}\left[E_r + \frac{1}{c}\left(v_{\alpha0\perp}B_\| - v_{\alpha0\|}B_\perp + v_{\alpha\perp}B_0\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&(-i\omega + i\mathbf{k}\cdot\mathbf{v}_{\alpha0})\, v_{\alpha\perp} + v_{\alpha r}\left(v'_{\alpha0\perp} + v_{\alpha0\|}\mathbf{e}_\perp\cdot\mathbf{h}' + \frac{h_z}{r}v_{\alpha0\vartheta}\right) + ik_\perp\left(\frac{T_\alpha}{m_\alpha} + \frac{T_{\alpha0}}{m_\alpha}\frac{n_\alpha}{n_{\alpha0}}\right) \\
&- \frac{1}{n_\alpha m_\alpha}F_{\alpha\perp} = \frac{e_\alpha}{m_\alpha}\left[E_\perp + \frac{1}{c}\left(v_{\alpha0\|}B_r - v_{\alpha r}B_0\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&(-i\omega + i\mathbf{k}\cdot\mathbf{v}_{\alpha0})\, v_{\alpha\|} + v_{\alpha r}\left(v'_{\alpha0\|} - v_{\alpha0\perp}\mathbf{e}_\perp\cdot\mathbf{h}' + \frac{h_\vartheta}{r}v_{\alpha0\vartheta}\right) + ik_\|\left(\frac{T_\alpha}{m_\alpha} + \frac{T_{\alpha0}}{m_\alpha}\frac{n_\alpha}{n_{\alpha0}}\right) \\
&- \frac{1}{n_\alpha m_\alpha}F_{\alpha\|} - \frac{1}{n_\alpha m_\alpha}R_{\alpha\|} = \frac{e_\alpha}{m_\alpha}\left[E_\| - \frac{1}{c}v_{\alpha0\perp}B_r\right],
\end{aligned}
$$

$$
(-i\omega + i\mathbf{k}\cdot\mathbf{v}_{\alpha0})\, n_\alpha + \left(n'_{\alpha0} + \frac{n_\alpha}{r}\right)v_{\alpha r} + n_{\alpha0}v'_{\alpha r} + i\mathbf{k}\cdot\mathbf{v}_\alpha n_{\alpha0} = 0,
$$

$$
\begin{aligned}
&\frac{3}{2}(-i\omega + i\mathbf{k}\cdot\mathbf{v}_{\alpha0})\frac{T_\alpha}{m_\alpha} + \frac{3}{2}v_{\alpha r}\frac{T'_{\alpha0}}{m_\alpha} + \frac{T_{\alpha0}}{m_\alpha}\left(v'_{\alpha r} + \frac{1}{r}v_{\alpha r} + i\mathbf{k}\cdot\mathbf{v}_\alpha\right) \\
&+ ik_\|\, q_{\alpha\|} + \Pi^{ij}\frac{\partial v_i}{\partial x^j} = Q_\alpha,
\end{aligned}
$$

$$
\begin{aligned}
&F^i = -\frac{\partial}{\partial x^j}\Pi^{ij}, \quad \Pi^{ij} = -\eta_\alpha^0 W_0^{ij} + \eta_\alpha^3 W_3^{ij} + \eta_\alpha^4 W_4^{ij}, \\
&R_{\alpha\|} = \mathrm{sgn}(e_\alpha)\left[-\frac{m_e n_e}{\tau_e}\, 0.51\,\mathbf{h}\cdot\mathbf{u} - 0.71\, n_e\,\mathbf{h}\cdot\nabla T_e\right], \quad \mathbf{u} = \mathbf{v}_e - \mathbf{v}_i, \\
&q_{e\|} = 0.71\, n_e\, T_e\, u_\| - 3.16\,\frac{n_e T_e \tau_e}{m_e}\,\mathbf{h}\cdot\nabla T_e, \quad q_{e\|} = -3.9\,\frac{n_i T_i \tau_i}{m_i}\,\mathbf{h}\cdot\nabla T_i.
\end{aligned}
$$

By angle averaging, $\langle A(\vartheta,\varphi)\rangle \overset{\mathrm{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\vartheta \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi\, A(\vartheta,\varphi)$, the quasilinear equations are obtained as,

$$
\frac{\partial}{\partial t}n_{\alpha0} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\langle n_\alpha v_{\alpha r}\rangle\right] = 0,
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}(m_\alpha n_{\alpha0}\mathbf{v}_{\alpha0}) + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left\langle m_\alpha n_\alpha \mathbf{v}_\alpha v_{\alpha r} + n_\alpha T_\alpha \mathbf{e}_r + \Pi^{rj}\mathbf{e}_j\right\rangle\right] \\
&= \left\langle e_\alpha n_\alpha\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}\right) \mp \mathbf{h}\, R_{\alpha\|}\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{2}m_\alpha n_{\alpha0}\mathbf{v}_{\alpha0}\cdot\mathbf{v}_{\alpha0} + \frac{3}{2}n_{\alpha0}T_{\alpha0}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left\langle\left(\frac{1}{2}m_\alpha n_\alpha\mathbf{v}_\alpha\cdot\mathbf{v}_\alpha + \frac{5}{2}n_\alpha T_\alpha\right)v_{\alpha r} + \Pi^{rj}v_{\alpha j}\right\rangle\right] \\
&= \left\langle e_\alpha n_\alpha \mathbf{v}_\alpha\cdot\mathbf{E} \mp \mathbf{v}_\alpha\cdot\mathbf{h}\, R_{\alpha\|} + Q_\alpha\right\rangle,
\end{aligned}
$$

where Q_α denotes the heat acquired by species α through collisions with the other species.

Numerical Realization

In the linear problem, a coupled system of first order ordinary differential equations with varying coefficients in the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}r} = \mathbf{A}(r) \cdot \mathbf{u},$$

with appropriate boundary conditions at the center of the cylinder, at the wall and at the antenna location has to be integrated numerically. In general this is a stiff problem, i.e., one of the eigenmodes grows much faster than the others. As a consequence, due to finite numerical accuracy of the integration process, the corresponding polarizations vectors become aligned and one cannot solve the boundary conditions at the antenna any more. In order to overcome this problem, a complete set of linearly independent modes which satisfy the boundary conditions at the center of the torus is integrated towards the antenna. The integrator routine contains a Runge-Kutta core and an orthonormalization-rescaling algorithm, i.e., the solution vectors $\mathbf{u}_i$ are orthonormalized into the set $\mathbf{e}_i$ during integration in prescribed intervals,

$$\mathbf{e}_i = \frac{1}{\beta_i}\left[\mathbf{u}_i - \sum_{j=1}^{i-1} \gamma_{ij}\mathbf{e}_j\right], \quad \gamma_{ij} = \mathbf{u}_i \cdot \mathbf{e}_j^*, \quad \beta_i = \left\| \mathbf{u}_i - \sum_{j=1}^{i-1} \gamma_{ij}\mathbf{e}_j \right\|.$$

The same procedure is used to integrate from the wall towards the antenna. At the the location of the antenna the remaining set of boundary conditions is used to find the appropriate combination of the modes and the solution vectors are obtained from rescaling backwards to center and wall. A post-processor module calculates the remaining quantities of physical interest. The modular structure of the code allows its application to a wide range of problems with inherent stiffness.

Figures 1–4 demonstrate the integration-orthonormalization procedure for the particular case of a cold plasma (two waves) and a DED frequency of 1 MHz. In this case, there exists a monotonic increasing fast mode with a characteristic scale of 4.5 cm and an oscillating slow mode with a characteristic wave length of 0.22 cm. One can clearly see in Figure 2 that without reorthonormalization, the slow wave polarization vector components (e.g., B_ϑ) starts to grow with the fast mode scale and, as a result, one could not solve for the boundary conditions at the antenna. With reorthonormalization, the numerical solution cannot be distinguished from the analytical solution for the homogeneous cylinder (Figures 3–4).

Conclusions

It is shown that, in principle, the stiff linear problem can be solved as long as the wavelengths involved are in a reasonable scale to each other. However, in the cold plasma model and for low frequencies the wave length of the slow mode becomes unreasonably (numerically as well as physically) small and thermal effects have to be added to the fluid equations. Including friction and pressure forces in the parallel direction alone does not seem to cure the problem. On the other hand, additional terms in perpendicular directions will increase the number of modes. This problem is currently under study.

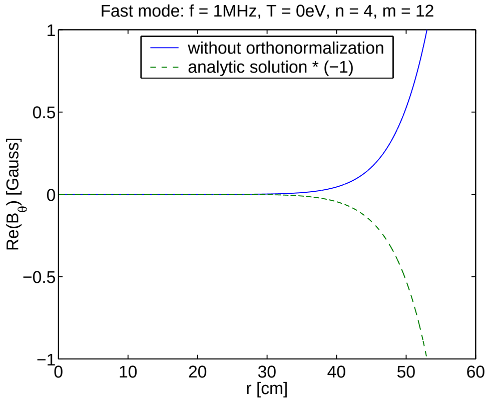

Fig. 1. The fast mode ($k_{rf} \approx i\frac{m}{r}\text{cm}^{-1}$) obtained numerically (solid line) and the analytical solution (dashed line) drawn with negative sign for comparison.

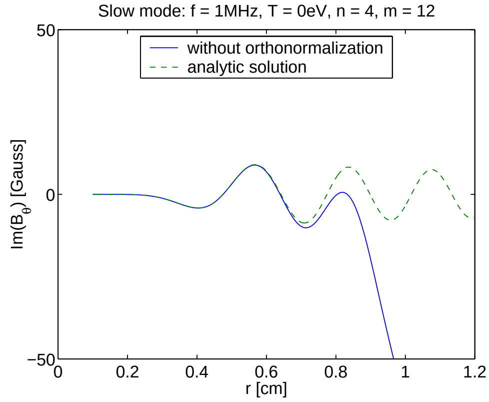

Fig. 2. The slow mode evolution obtained numerically without reorthonormalization (solid line) and the analytical solution (dashed line). After a couple of oscillations the numerical solution starts to grow similar to the fast mode due to finite numerical precision.

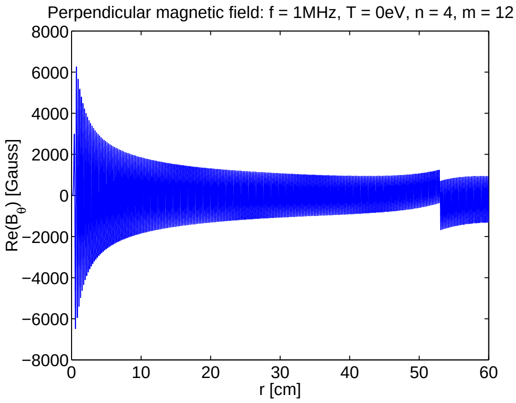

Fig. 3. Radial profile of the perpendicular magnetic magnetic field Re(B_ϑ) after solving for the boundary conditions at the antenna location, r=53 cm. The discontinuous behavior of B_ϑ and B_z corresponds to the prescribed (Fourier decomposed) antenna currents.

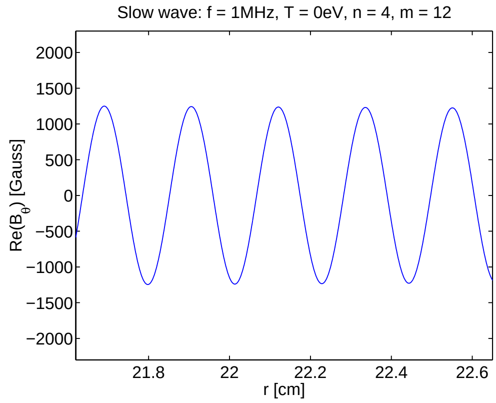

Fig. 4. Using the orthonormalization algorithm one can actually resolve the slow mode behavior with wave length $\lambda_{rs} \approx 0.22$cm (detail of Figure 3).

References

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