

THE TRANSPORT COEFFICIENTS OF A FUSION PLASMA ON AN INTERMEDIATE MHD TIMESCALE

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1. INTRODUCTION

In [1] the general theory of a new multiple timescale (MTS) derivative expansion scheme has been presented and applied both to the dimensionless Fokker-Planck and to Maxwell's equations. Within kinetic theory, the four species dependent timescales are those of Larmor gyration Ω_α^{-1} , the transit time ω_α^{-1} , the collision time ν_α^{-1} and the classical diffusion time $\tau_{\alpha cd}$, which can be written in the standardized form $\tau_{\alpha n} = \tau_{\alpha 0} \delta_\alpha^{-n}$, where $\delta_\alpha = \omega_\alpha / \Omega_\alpha$. The application of the MTS approach to the dimensionless Fokker-Planck equation leads to a separate kinetic equation for each order in the expansion parameter δ_α .

2. BASIC EQUATIONS

The laboratory frame representation of the kinetic equations is appropriate for deriving the transport equations via velocity moments, i.e. for the analysis of the convective transport. However, for the investigation of the diffusive transport, i.e. for the calculation of the transport coefficients depending only on the random velocity, we prefer to represent the kinetic equations in the convected velocity reference (CVR) frame. For that we have to rewrite the hierarchy of Fokker-Planck equations (Eqs. (8) in Ref. [1]).

Replacing the velocity \vec{v} by the random velocity $\vec{c} = \vec{v} - \vec{u}_\alpha$, the dimensionless Fokker-Planck equation in the CVR frame reads [3]

$$\delta_\alpha \left(\frac{df_\alpha}{dt_\alpha} + \vec{c} \cdot \nabla f_\alpha \right) + \left\{ \sigma_\alpha (\vec{F}_\alpha + \vec{c} \times \vec{B}) - \delta_\alpha \vec{c} \cdot \mathbf{S}_\alpha - \delta_\alpha \frac{d\vec{u}_\alpha}{dt_\alpha} \right\} \cdot \nabla_{\vec{c}} f_\alpha = \delta_\alpha^2 \Lambda_\alpha C_\alpha(f_\alpha, f_\beta), \quad (1)$$

with the definitions

$$\frac{d}{dt_\alpha} := \frac{\partial}{\partial t} + \vec{u}_\alpha \cdot \nabla, \quad \sigma_\alpha := \text{sign}(q_\alpha), \quad \vec{F}_\alpha := a_\alpha \vec{E} + \vec{u}_\alpha \times \vec{B}, \quad \text{and} \quad \mathbf{S}_\alpha := \nabla \otimes \vec{u}_\alpha.$$

Application of the MTS approach leads to the following zeroth-order and first-order equations in the CVR frame:

$$\delta_\alpha^0 : \sigma_\alpha (\vec{F}_{\alpha 0} + \vec{c} \times \vec{B}_0) \cdot \nabla_{\vec{c}} f_{\alpha 0} = 0 \quad (2)$$

$$\begin{aligned} \delta_\alpha^1 : & \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 0}} + \vec{u}_{\alpha 0} \cdot \nabla f_{\alpha 0} + \vec{c} \cdot \nabla f_{\alpha 0} + \sigma_\alpha (\vec{F}_{\alpha 0} + \vec{c} \times \vec{B}_0) \cdot \nabla_{\vec{c}} f_{\alpha 1} \\ & + \left\{ \sigma_\alpha \left(\vec{F}_{\alpha 1} + \left(\frac{\delta_\beta}{\delta_\alpha} \right)^{\frac{1}{2}} \vec{c} \times \vec{B}_1 \right) - \vec{c} \cdot \mathbf{S}_{\alpha 0} - \frac{\partial \vec{u}_{\alpha 0}}{\partial t_{\alpha 0}} - (\vec{u}_{\alpha 0} \cdot \nabla) \vec{u}_{\alpha 0} \right\} \cdot \nabla_{\vec{c}} f_{\alpha 0} = 0 \end{aligned} \quad (3)$$

where $\vec{F}_{\alpha 0} := a_\alpha \vec{E}_0 + \vec{u}_{\alpha 0} \times \vec{B}_0$ and $\vec{F}_{\alpha 1} := \vec{u}_{\alpha 1} \times \vec{B}_0 + \left(\frac{\delta_\beta}{\delta_\alpha} \right)^{\frac{1}{2}} (a_\alpha \vec{E}_1 + \vec{u}_{\alpha 0} \times \vec{B}_1)$. Using the zeroth-order and first-order two-fluid transport equations derived in the same way as in [1], the forces $\vec{F}_{\alpha 0}$ and $\vec{F}_{\alpha 1}$ can be eliminated finally yielding

$$\delta_\alpha^0 : \sigma_\alpha (\vec{c} \times \vec{B}_0) \cdot \nabla_{\vec{c}} f_{\alpha 0} = 0 \quad (4)$$

$$\delta_\alpha^1 : \frac{df_{\alpha 0}}{dt_{\alpha 0}} + \vec{c} \cdot \nabla f_{\alpha 0} + \left(\frac{\nabla p_{\alpha 0}}{n_{\alpha 0}} - \vec{c} \cdot \mathbf{S}_{\alpha 0} \right) \cdot \nabla_{\vec{c}} f_{\alpha 0} + \sigma_\alpha (\vec{c} \times \vec{B}_0) \cdot \nabla_{\vec{c}} f_{\alpha 1} = 0 \quad (5)$$

with the solution of the zeroth-order equation, Eq. (4), simply given by

$$f_{\alpha 0}(\vec{x}, \vec{c}, t_{\alpha 0}, t_{\alpha 1}, t_{\alpha 2}) = n_{\alpha 0} (\pi T_{\alpha 0})^{-3/2} \exp(-\vec{c}^2 / T_{\alpha 0}). \quad (6)$$

3. FIRST-ORDER SOLUTION $f_{\alpha 1}$

In order to solve the first-order equation, Eq. (5), we assume that $f_{\alpha 1}$ is only slightly deviating from $f_{\alpha 0}$ and engage the ansatz

$$f_{\alpha 1} = \Phi_\alpha(\vec{x}, \vec{c}, t) f_{\alpha 0}, \quad (7)$$

where Φ_α denotes a small correction. Thus we arrive at the first-order Fokker-Planck equation of the form

$$-\sigma_\alpha (\vec{c} \times \vec{B}_0) \cdot \nabla_{\vec{c}} \Phi_\alpha = \left(\frac{\vec{c}^2}{T_{\alpha 0}} - \frac{5}{2} \right) \vec{c} \cdot \nabla (\ln T_{\alpha 0}) + \frac{2}{T_{\alpha 0}} \vec{c} \cdot \mathbf{e}_{\alpha 0} \cdot \vec{c} \quad (8)$$

with $\mathbf{e}_{\alpha 0} := \frac{1}{2} (\mathbf{S}_{\alpha 0} + \mathbf{S}_{\alpha 0}^T) - \frac{1}{3} \text{tr}(\mathbf{S}_{\alpha 0}) \mathbf{I}$. This equation is an inhomogeneous, linear first-order differential equation for Φ_α . The conditions for the existence of a solution are $\vec{b} \cdot \nabla T_{\alpha 0} = 0$ and $\vec{b} \cdot \mathbf{e}_{\alpha 0} \cdot \vec{b} = 0$, with $\vec{b} = \vec{B}_0 / B_0$, $B_0 = |\vec{B}_0|$. For solving Eq. (8) we express Φ_α as

$$\Phi_\alpha(\vec{x}, \vec{c}, t) = \Phi_\alpha^{\text{hom}} + \Phi_\alpha^{\text{inh}} = \Phi_\alpha^{\text{hom}} + \vec{c} \cdot \vec{U}_\alpha + \vec{c} \cdot \mathbf{V}_\alpha \cdot \vec{c} \quad (9)$$

where Φ_α^{hom} and Φ_α^{inh} refer to the homogeneous and inhomogeneous solutions, respectively.

Evidently, the non-trivial homogeneous solution is $\Phi_\alpha^{\text{hom}} = F(c_\parallel, c; \vec{x}, t)$ with $c_\parallel = \vec{b} \cdot \vec{c}$ and $c = |\vec{c}|$, where F is an arbitrary function which may be represented as a double series

$$\Phi_\alpha^{\text{hom}} = F = \sum_m \sum_n b_{m,n}^\alpha c_\parallel^m c^n. \quad (10)$$

For \vec{U}_α and \mathbf{V}_α we found

$$\vec{U}_\alpha = \frac{\sigma_\alpha}{B_0 T_{\alpha 0}} \left(\frac{c^2}{T_{\alpha 0}} - \frac{5}{2} \right) \vec{b} \times \nabla T_{\alpha 0} \text{ and } \mathbf{V}_\alpha = \frac{\sigma_\alpha}{B_0 T_{\alpha 0}} \begin{bmatrix} -\mathbf{e}_{\alpha 0}^{12} & \frac{1}{2}(\mathbf{e}_{\alpha 0}^{11} - \mathbf{e}_{\alpha 0}^{22}) & -2\mathbf{e}_{\alpha 0}^{23} \\ \frac{1}{2}(\mathbf{e}_{\alpha 0}^{11} - \mathbf{e}_{\alpha 0}^{22}) & \mathbf{e}_{\alpha 0}^{12} & 2\mathbf{e}_{\alpha 0}^{13} \\ -2\mathbf{e}_{\alpha 0}^{23} & 2\mathbf{e}_{\alpha 0}^{13} & 0 \end{bmatrix}. \quad (11)$$

A local Cartesian coordinate system has been chosen for representing \mathbf{V}_α , in which the x_3 -axis is aligned along the zeroth-order magnetic field \vec{B}_0 . Thus we found the general solution for the first-order distribution function $f_{\alpha 1}$ to be

$$f_{\alpha 1} = n_{\alpha 0} (\pi T_{\alpha 0})^{-3/2} \left[\sum_m \sum_n b_{m,n}^\alpha c_\parallel^m c^n + \vec{c} \cdot \vec{U}_\alpha + \vec{c} \cdot \mathbf{V}_\alpha \cdot \vec{c} \right] \exp(-\vec{c}^2 / T_{\alpha 0}) \quad (12)$$

where the coefficients $b_{m,n}^\alpha$ are consistent with the requirement $\int \vec{c} f_{\alpha 1} d\vec{c} = 0$.

4. HEAT FLUX AND VISCOSITY

Introducing Eq. (12) into the definition for the dimensionless first-order heat flux,

$\vec{q}_{\alpha 1} = \frac{1}{2} \int c^2 \vec{c} f_{\alpha 1} d\vec{c}$, we arrive at

$$\vec{q}_{\alpha 1} = \sum_{j=0} \sum_{k=0} \frac{n_{\alpha 0} T_{\alpha 0}^{j+k+2}}{2j+3} \left\{ \frac{[2(j+k)+5]!!}{2^{j+k+3}} b_{2j+1, 2k}^\alpha + (j+k+3)! \sqrt{\frac{T_{\alpha 0}}{\pi}} b_{2j+1, 2k+1}^\alpha \right\} \vec{b} + \frac{5}{8} \frac{\sigma_\alpha n_{\alpha 0} T_{\alpha 0}}{B_0} \vec{b} \times \nabla T_{\alpha 0} \quad (13)$$

with the dimensional representation of the second term (originating from Φ_α^{inh})

$$\vec{q}_\alpha^{\text{inh}} = \kappa_\alpha (\vec{b} \times \tilde{\nabla} \tilde{T}_{\alpha 0}) \text{ with } \kappa_\alpha = \frac{5}{2} \frac{\sigma_\alpha \tilde{n}_{\alpha 0} \tilde{T}_{\alpha 0}}{m_\alpha \Omega_{\alpha 0}}, \quad (14)$$

where the over-tilde denotes the dimensional physical quantities. The thermal conductivities κ_α and 0, related to the temperature gradients $\vec{b} \times \tilde{\nabla} \tilde{T}_{\alpha 0}$ and $\tilde{\nabla}_\perp \tilde{T}_{\alpha 0}$, are identical to Braginskii's corresponding coefficients κ_α^α and κ_\perp^α [2] in the collisionless limit.

From the first-order viscosity tensor definition $\vec{\Pi}_{\alpha 1} = \int \vec{c} \otimes \vec{c} f_{\alpha 1} d\vec{c} - \frac{\mathbf{I}}{3} \int c^2 f_{\alpha 1} d\vec{c}$ we find

$$\tilde{\Pi}_{\alpha 1} = \sum_{j=0} \sum_{k=0} \frac{4jn_{\alpha 0} T_{\alpha 0}^{j+k+1}}{(2j+1)(2j+3)} \left\{ \frac{[2(j+k)+3]!!}{2^{j+k+2}} b_{2j,2k}^{\alpha} + (j+k+2)! \sqrt{\frac{T_{\alpha 0}}{\pi}} b_{2j,2k+1}^{\alpha} \right\} \left(\vec{b} \otimes \vec{b} - \frac{\mathbf{I}}{3} \right) + \frac{n_{\alpha 0} T_{\alpha 0}^2}{2} \mathbf{V}_{\alpha}. \quad (15)$$

Here the dimensional representation of the second term (originating from $\Phi_{\alpha}^{\text{inh}}$) is

$$\tilde{\Pi}_{\alpha}^{\text{inh}} = \eta_{\alpha} \begin{bmatrix} -\tilde{\mathbf{e}}_{\alpha 0}^{12} & \frac{1}{2}(\tilde{\mathbf{e}}_{\alpha 0}^{11} - \tilde{\mathbf{e}}_{\alpha 0}^{22}) & -2\tilde{\mathbf{e}}_{\alpha 0}^{23} \\ \frac{1}{2}(\tilde{\mathbf{e}}_{\alpha 0}^{11} - \tilde{\mathbf{e}}_{\alpha 0}^{22}) & \tilde{\mathbf{e}}_{\alpha 0}^{12} & 2\tilde{\mathbf{e}}_{\alpha 0}^{13} \\ -2\tilde{\mathbf{e}}_{\alpha 0}^{23} & 2\tilde{\mathbf{e}}_{\alpha 0}^{13} & 0 \end{bmatrix} \quad \text{with} \quad \eta_{\alpha} = \frac{\sigma_{\alpha} \tilde{n}_{\alpha 0} \tilde{T}_{\alpha 0}}{\Omega_{\alpha 0}}. \quad (16)$$

The viscosity coefficient η_{α} is the collisionless limit of Braginskii's [2] corresponding viscosity coefficients η_4^{α} and $2\eta_3^{\alpha}$. The collisionless limits of Braginskii's η_1^{α} and η_2^{α} are zero; accordingly, there is no corresponding contribution to our $\tilde{\Pi}_{\alpha}^{\text{inh}}$.

5. SUMMARY

Based on the application of a MTS approach to the Fokker-Planck equation and to Maxwell's equations, the particle heat flux and the viscosity tensor were derived. Braginskii's parallel heat flux related to $\tilde{\nabla}_{\parallel} \tilde{T}_{\alpha}$ is undefined in the collisionless limit, as is the collisionless limit of Braginskii's viscosity related to $\tilde{\mathbf{e}}_{\alpha}^{33}$. These contributions to the heat flux and to the viscosity tensor are replaced in our approach by the contributions originating from $\Phi_{\alpha}^{\text{hom}}$. Apart from these contributions, the heat flux and the viscosity tensor obtained here are equivalent to the collisionless limits of Braginskii's results.

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Obituary

This contribution was completed in memoriam of our highly respected colleague Prof. Dr. Johann Edenstrasser who deceased on February 4th 2001 due to a tragic accident. Not only having initiated the reported study, he was the leading scientist in this research.

Acknowledgement

The reported work has been partially carried out within the Association EURATOM-OEAW projects P1, P4 and P8 and within the impact project *Investigation of Charged Fusion Product Confinement in JET* funded by OEAW under GZ 4229/1-VIII/A/5/2000.