# Quasisymmetry far from the magnetic axis

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# Outline

## MOTIVATION

- Undamped flows in optimized stellarators: quasisymmetry (QS).
- Why we will focus on *approximate* QS.

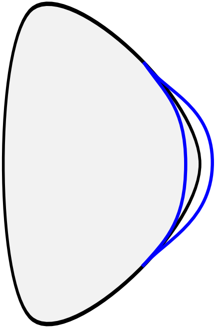
## AND THEN . . .

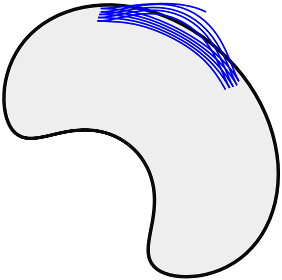
- **Part I:** When can a stellarator be considered quasisymmetric (QS) in practice?
  - Formal criterion for 'closeness to quasisymmetry'.
- **Part II:** What is the size of the radial region in which the criterion can be satisfied?

$$\mathbf{v}_{M,s} = \frac{v_{\parallel}^2}{\Omega_s}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}) + \frac{v_{\perp}^2}{2B\Omega_s}\hat{\mathbf{b}} \times \nabla B, \qquad v_{\psi,s} := \mathbf{v}_{M,s} \cdot \nabla\psi,$$

where $\Omega_s$ is the gyrofrequency of species $s$ and $\psi$ the toroidal flux.

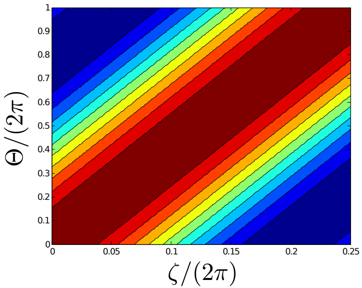
- In tokamaks, trapped particles are well confined: the radial magnetic drift averaged over the orbit, $\overline{v_{\psi,s}}$, vanishes.
- In general, in a stellarator, $\overline{v_{\psi,s}} \neq 0$ and trapped particles drift away.





# Optimized stellarators: omnigeneity and quasisymmetry

- A stellarator is called omnigeneous if $\overline{v_{\psi,s}} = 0$. [Cary and Shasharina (1997), Parra *et al.* (2015)].
- Flows are generally damped in an omnigeneous stellarator.
- A stellarator that possesses a direction in which flows are undamped is called quasisymmetric [Helander and Simakov (2008)].
    - Quasisymmetry implies omnigeneity.




- In a QS stellarator, there exist privileged sets of coordinates $\{\psi, \Theta, \zeta\}$, where $\Theta$ is a poloidal angle and $\zeta$ is a toroidal angle, such that $|B(\psi, \Theta, \zeta)|$ has a symmetry direction as shown in the figure. They are called Boozer coordinates.

# Why study QS stellarators

- The outer region of a stellarator plasma is dominated by turbulence. See, for example, [Dinklage *et al.* (2013)].
- Flow shear reduces turbulence.
- It seems reasonable to investigate configurations that admit large flows (i.e. such that flows are undamped) as a route to achieve large flow shear.

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**However, it only makes sense to consider stellarators close to quasisymmetry. . .**

- Exact quasisymmetry cannot be achieved throughout the entire plasma volume [Garren and Boozer (1991); expansion around the magnetic axis].

**Can we quantify closeness to quasisymmetry? When is flow damping sufficiently small?**

# Part I: When can a stellarator be considered QS in practice?

Take a divergenceless vector field of the form $\mathbf{Y} = B^{-1}\hat{\mathbf{b}} \times \nabla\psi + h\mathbf{B}$.

**Total flow damping in the direction $\mathbf{Y}$:**

$$\Big\langle (\nabla \cdot \boldsymbol{\pi}_i) \cdot \mathbf{Y} \Big\rangle_\psi = -\frac{1}{c}\Big\langle \mathbf{J}_n \cdot \nabla\psi \Big\rangle_\psi + \Big\langle (\nabla \cdot \boldsymbol{\pi}_{\mathrm{gy},i}) \cdot \mathbf{Y} \Big\rangle_\psi,$$

where $\boldsymbol{\pi}_i$ is the viscosity tensor, $\langle \mathbf{J}_n \cdot \nabla\psi \rangle_\psi$ is the neoclassical radial electric current density and $\boldsymbol{\pi}_{\mathrm{gy},i}$ is the gyroviscosity.

- $c^{-1}\langle \mathbf{J}_n \cdot \nabla\psi \rangle_\psi \sim \rho_{*i}^2 c^{-1} e n_i v_{ti} |\nabla\psi|$ is the largest term in a generic stellarator.
- $\Big\langle (\nabla \cdot \boldsymbol{\pi}_{\mathrm{gy},i}) \cdot \mathbf{Y} \Big\rangle_\psi \sim \rho_{*i}^3 c^{-1} e n_i v_{ti} |\nabla\psi|$ includes turbulent and higher-order neoclassical contributions.

Here, $c$ is the speed of light, $e$ is the proton charge, $v_{ti}$ is the ion thermal speed, $n_i$ is the ion density, and $\rho_{*i}$ is the ion Larmor radius over the major radius, $R_0$.

# Part I: When can a stellarator be considered QS in practice?

$$\left\langle (\nabla\cdot\boldsymbol{\pi}_i)\cdot\mathbf{Y}\right\rangle_\psi = \left[\rho_{*i}^2 A_{\rm nc} + \rho_{*i}^3 A_{\rm gy}\right] c^{-1} e n_i v_{ti} |\nabla\psi| + \dots$$

- $A_{\rm nc}$ and $A_{\rm gy}$ are $O(1)$, generically.
- $A_{\rm nc} \equiv 0$ **if and only if the stellarator is quasisymmetric.**
- **The idea:** Take $\mathbf{B} = \mathbf{B}_0 + \alpha\mathbf{B}_1$, where $\mathbf{B}_0$ is quasisymmetric and $0 < \alpha \ll 1$.
  - We expect $A_{\rm nc} \sim \alpha^q \nu_{*i}^r \overline{A}_{\rm nc}$, with $\overline{A}_{\rm nc} = O(1)$ and $\nu_{*i}$ the ion collisionality.
  - Compare $\rho_{*i}^2 \alpha^q \nu_{*i}^r$ with $\rho_{*i}^3$ to obtain the criterion.
- The stellarator can be considered quasisymmetric in practice if

$$\alpha \ll \left(\rho_{*i}\nu_{*i}^{-r}\right)^{1/q}.$$

The scalings depend on the collisionality regime (obvious) and on the size of the gradients of $B_1$ (not so obvious).

# Perturbations with small gradients [Calvo, Parra, Velasco and Alonso (2013)]

- $\mathbf{B} = \mathbf{B}_0 + \alpha \mathbf{B}_1$, where $\alpha \mathbf{B}_1$ is a small deviation from quasisymmetry.
- If

$$|\alpha \partial_\Theta B_1| / |\partial_\Theta B_0| \sim \alpha \quad \text{and} \quad |\alpha \partial_\zeta B_1| / |\partial_\zeta B_0| \sim \alpha,$$

  then the drift-kinetic equation and $\langle \mathbf{J}_n \cdot \nabla \psi \rangle_\psi$ can be Taylor expanded, and it can be proven that

$$\Big\langle (\nabla \cdot \boldsymbol{\pi}_i) \cdot \mathbf{Y} \Big\rangle_\psi = \Big[ \rho_{*i}^2 \alpha^2 \, \hat{A}_{\text{nc}} + \rho_{*i}^3 A_{\text{gy}} \Big] c^{-1} e n_i v_{ti} |\nabla \psi| + \dots .$$

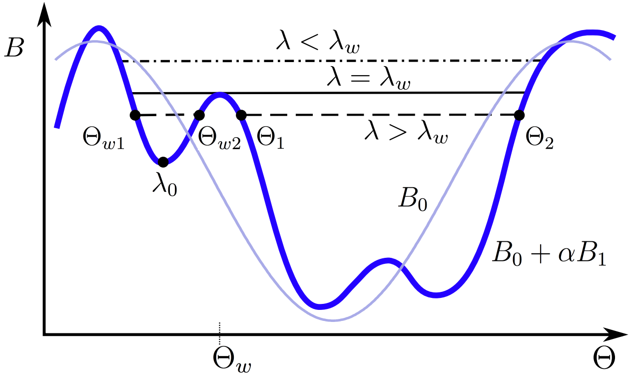
- True for any collisionality regime but, of course, the size of $\hat{A}_{\text{nc}}$ depends on $\nu_{*i}$. **In the $1/\nu$ regime** we obtain the criterion

$$\alpha \ll \sqrt{\nu_{*i} \rho_{*i}}$$

  for closeness to quasisymmetry.

# Perturbations with large gradients [Calvo, Parra, Alonso and Velasco (2014)]

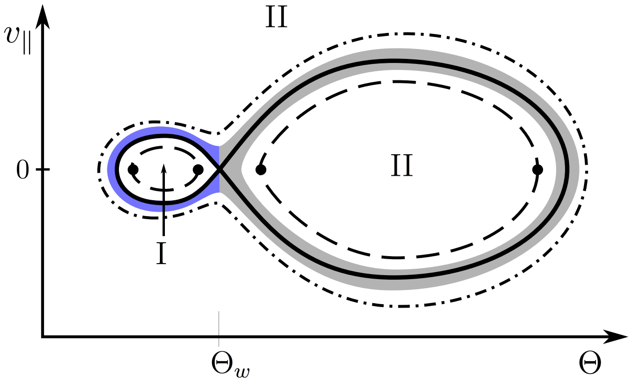
- If $|\alpha\partial_\Theta B_1|/|\partial_\Theta B_0| \sim 1$ or $|\alpha\partial_\zeta B_1|/|\partial_\zeta B_0| \sim 1$, then a simple Taylor expansion does not work.




**We have solved the $1/\nu$ regime. Typical passing and trapped (both in large wells and ripple wells) particles are collisionless**.

# Perturbations with large gradients: $1/\nu$ regime

- The drift-kinetic equation must be solved in each of the regions numbered in the figure, and the solutions have to be matched.
- In contrast to the common perception, **the neoclassical fluxes and therefore the neoclassical damping are not dominated by ripple wells**.



# Region II, particles trapped in large wells, dominate neoclassical transport for $\nu_{*i} \ll 1$

**The key result is the determination of the scaling of the orbit-averaged radial magnetic drift, $\overline{v_{\psi,i}}$.**

- The region of the orbit in a neighborhood of the bounce points is responsible for the result $\overline{v_{\psi,i}} \sim \alpha^{1/2}\rho_{*i}v_{ti}|\nabla\psi|$. Relatively technical calculation [Calvo *et al.* (2014)].
- The non-adiabatic piece of the distribution function in this region scales as $G_i^{\mathrm{II}} \sim \alpha^{1/2}\nu_{*i}^{-1}\rho_{*i}n_i v_{ti}^{-3}$, and large wells contribute to the damping as

$$\langle \mathbf{J}_n \cdot \nabla\psi \rangle_\psi \sim \frac{\alpha}{\nu_{*i}}\rho_{*i}^2 n_i v_{ti}|\nabla\psi|.$$

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- The criterion for closeness to quasisymmetry is, in this case,

$$\alpha \ll \nu_{*i}\rho_{*i}.$$

- Compare with the less demanding criterion obtained for perturbations with small gradients, $\alpha \ll \sqrt{\nu_{*i}\rho_{*i}}$.

# Part II: What is the size of the radial region in which the criterion for closeness to QS can be satisfied?

- We have to understand the relation between magnetohydrodynamic (MHD) equilibrium equations and the QS condition.
  - This is done in the basic reference [Garren and Boozer (1991)], relying on an expansion around the magnetic axis.

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## MHD EQUILIBRIUM AROUND A SURFACE $\mathcal{S}_{r_0}$ FAR FROM THE MAGNETIC AXIS

- Data that are needed on $\mathcal{S}_{r_0}$ to determine $\mathbf{B}$ around $\mathcal{S}_{r_0}$?
  - Here, $r := \sqrt{\psi/B_{\rm axis}}$, where $B_{\rm axis}$ is the average of $B$ on axis.
- **We have developed a method (not explained in detail here) to compute local MHD equilibria in arbitrary flux coordinates $\{r, u, v\}$, where $u$ and $v$ are poloidal and toroidal angles.**
  - In [Hegna (2000)], the problem is addressed in Boozer coordinates.

# Local stellarator MHD equilibria

If $\mathbf{B} \cdot \nabla r = 0$ and $(\nabla \times \mathbf{B}) \cdot \nabla r = 0$, then

$$\mathbf{B} = I_t(r) \nabla_{\mathcal{S}_r} u + I_p(r) \nabla_{\mathcal{S}_r} v + \nabla_{\mathcal{S}_r} \chi$$

where $\nabla_{\mathcal{S}_r}$ denotes the projection of the gradient on $\mathcal{S}_r$. **The potential $\chi(r, u, v)$ depends on the choice of coordinates**.

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Using $\nabla \cdot \mathbf{B} = 0$ and the MHD equilibrium equations, one can prove that

- Knowledge of $\mathbf{x}(r_0, u, v)$ (i.e. the shape of $\mathcal{S}_{r_0}$) and two numbers (say, $I_t(r_0)$ and $I_p(r_0)$) determines $\mathbf{B}$ on $\mathcal{S}_{r_0}$.
- Knowledge of $\mathbf{x}(r_0, u, v)$ and two flux functions (say, $I_t(r)$ and $I_p(r)$) determines $\mathbf{B}$ on $\mathcal{S}_{r_0+\delta r}$ and, by integration, in the whole stellarator.
- Note that, out of the three functions of $u$ and $v$ within $\mathbf{x}(r_0, u, v)$, two of them correspond to reparameterizations on $\mathcal{S}_{r_0}$. **The real freedom amounts to only one function of $u$ and $v$**.

# Exact QS on $\mathcal{S}_{r_0}$ is possible

Coordinate-free QS condition. A stellarator is QS if and only if

$$(\mathbf{B} \times \nabla r) \cdot \nabla B = F(r)\mathbf{B} \cdot \nabla B$$

for some flux function $F(r)$.

Geometrically, a stellarator is QS if there exists $F(r)$ such that the vector field $\mathbf{V}_F := \mathbf{B} \times \nabla r - F(r)\mathbf{B}$ has closed integral curves and such that $B$ is constant along them; i.e.

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Express the QS condition in terms of $\mathbf{x}(r_0, u, v)$ by employing that

$$B^2 = \frac{(I_t + \partial_u \chi)^2 |\partial_v \mathbf{x}|^2}{|\partial_u \mathbf{x} \times \partial_v \mathbf{x}|^2} + \frac{(I_p + \partial_v \chi)^2 |\partial_u \mathbf{x}|^2}{|\partial_u \mathbf{x} \times \partial_v \mathbf{x}|^2} - 2(I_t + \partial_u \chi)(I_p + \partial_v \chi) \frac{\partial_u \mathbf{x} \cdot \partial_v \mathbf{x}}{|\partial_u \mathbf{x} \times \partial_v \mathbf{x}|^2},$$

etc.

**The QS condition provides an additional equation that fixes the function of $u$ and $v$ that remained free after imposing MHD equilibrium.**

# Deviation from quasisymmetry around $\mathcal{S}_{r_0}$

- Since **we have no more freedom after making $\mathcal{S}_{r_0}$ exactly QS**, $\mathbf{B}$ around $\mathcal{S}_{r_0}$ is found from the local equilibrium relations.
- Schematically,

$$\mathbf{x}(r_0 + \delta r, u, v) = \mathbf{x}(r_0, u, v) + \partial_r \mathbf{x}(r_0, u, v)\delta r + \dots$$

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- Typically, within a radial distance $\delta r$ from $\mathcal{S}_{r_0}$, deviations from QS of size

$$\alpha \sim \delta r / a$$

are generated. Here, $a$ is the minor radius. Recalling the most favorable criterion for closeness to QS (small gradients and $\nu_* \sim 1$),

$$\alpha \ll \rho_{*i}^{1/2},$$

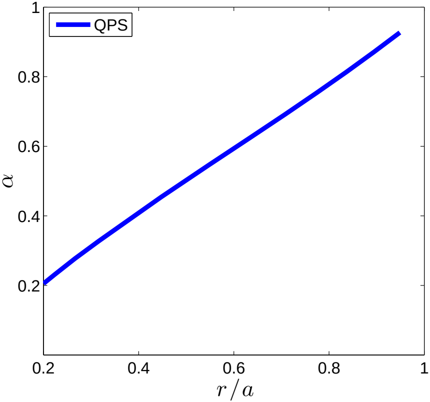
we estimate that the stellarator can be QS in practice in a region

$$\delta r / a \ll \rho_{*i}^{1/2}.$$

**This looks like a quite negative result...**

# This negative result seems to be consistent when we look at the configuration of relatively compact stellarators

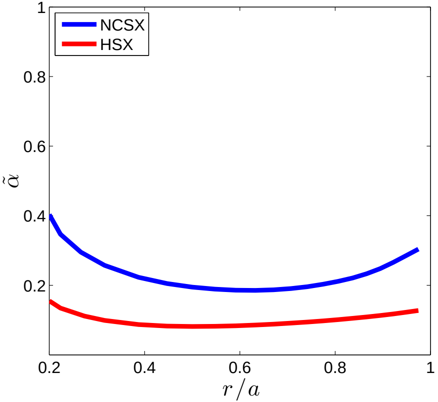
- QPS has an aspect ratio $\epsilon^{-1} := R_0/a \approx 2.7$.




- Here, $\tilde{\alpha} := \alpha/\epsilon(r)$ is the suitable quantity to measure deviations from QS when $\epsilon(r) := r/R_0 \ll 1$.

# But things work better for large aspect ratio stellarators

- It is possible to achieve $\tilde{\alpha} \ll 1$ in a considerable radial region.




- NCSX, $\epsilon^{-1} := R_0/a \approx 4.6$.
- HSX, $\epsilon^{-1} := R_0/a \approx 10$.

- **We understand this only partially.**

- Starting with an exactly QS magnetic axis, [Garren and Boozer (1991)] proved that $\epsilon(r) << 1$ allowed QS in its neighborhood to high accuracy.
- Our objective was to make a generic surface $\mathcal{S}_{r_0}$ exactly QS, and estimate deviations from QS around it.
- **For $\epsilon \sim 1$, we understand the problem. For $\epsilon \ll 1$, work to be done.**

# Thank you for your attention!

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