EUROfusion

Predicting the stability of alpha-particle-driven Alfvén Eigenmodes in burning plasmas

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Acknowledgements:

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α-particles in fusion plasmas.

Energetic α-particles produced in nuclear fusion reactions are a key ingredient to a ignited plasma able to produce energy.
[Fasoli 2007]

During the burning regime in fusion reactors:

- Isotropic fusion-born αs provide the main plasma heating;
- They need to be kept confined in the core;
- Their energy must be transferred to the bulk plasma;
- They must be prevented from reaching the walls;

What can go wrong?

In fusion plasmas, α-particles are near-Alfvénic:

- 3.5 MeV αs have $v_{||} \sim 10^7$ m/s;
- The Alfvén velocity in ITER is about $v_A \sim 7 \times 10^6$ m/s;

Alfvén Eigenmodes (AEs) can be destabilized:

- AEs are driven by resonant energy transfer from α-particles;
- Ustable AEs may redistribute α-particles away from the plasma core and towards the walls;

What needs to be done:

Develop predictive capability to understand the interaction of α-particles with AEs and their stability in burning plasmas;

- Handle routine stability assessments and sensitivity analysis;
- Guide experiment planning and design;

Outline.

1. Systematic approach to the stability of AEs in fusion plasmas;
   - Handle routine stability assessments and sensitivity analysis;
   - Guide experiment planning and design;
2. Stability assessment of ITER's $I_{\mathrm{p}} = 15$ MA baseline scenario;
   - Identify the most unstable AEs;
   - Discuss their properties;
3. Sensitivity analysis of ITER's $I_{\mathrm{p}} = 15$ MA baseline scenario;
   - Slightly change the background magnetic equilibrium;
   - Evaluate and discuss the changes caused in stability properties;
4. Discuss properties of the wave-particle resonant interaction;
   - Distinct energy-transfer efficiency for resonant orbits;
   - Drift-velocity effects on the resonance condition;
5. Summary and conclusions.

Predictive modelling.

Making predictions for burning plasmas with a non-thermal α-particle population is a complex and demanding task.

When designing and planning experiments...

- Multiple scenarios and configurations need to be considered;
- The AEs most easily destabilized in each one must be found.

One solution to the problem:

- Scan the space $(\omega, \mathbf{k})$ to find all possible AEs for a given magnetic equilibrium;
- Assess the linear stability of the whole set;

Major aim:

Guide experiment planning and design by identifying the most-relevant AEs for later analysis with more detailed tools.

Particle-wave interaction model.

Comprehensive models:

First-principles approach (e.g., nonlinear gyrokinetic);

- Computationally demanding;
- Not suitable for routine stability assessments.

Linear hybrid MHD–drift-kinetic model:

1. Scan the frequency and toroidal-n ranges with the ideal-MHD code `MISHKA` [Mikhailovskii 1997];
2. Evaluate the energy exchange between AEs and each species (αs, DT, e^-, He ash) with `CASTOR-K` [Borba 1999, Nabais 2015].

List of possible AEs sorted by growth (or damping) rate.

The Alfvén Stability Package.

- Front-end to several numerical codes used in predictive modelling of AEs in burning plasmas;
- Able to efficiently handle routine stability assessments and sensitivity analysis.

Hybrid model and code efficiency:

- Restricted to the linear stage of the particle-wave interaction;
- `MISHKA` and `CASTOR-K` are well optimized and tested;

Easy workload sharing and distribution:

- Take advantage of massively-parallel computers;
- Distribute along $(\omega, \mathbf{k})$-space subsets to be scanned;
- Distribute along each AE to be processed by `CASTOR-K`.

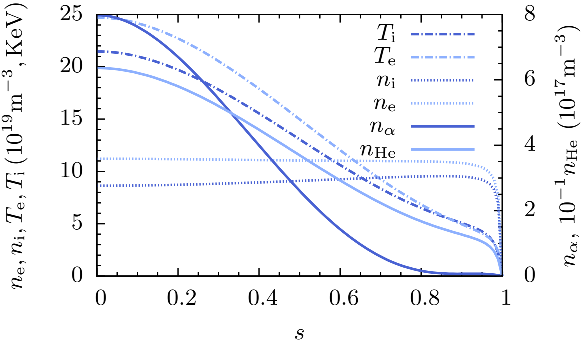
Partial summary I:

A systematic approach is able to handle routine stability assessments and sensitivity analysis in burning plasmas;

- Hybrid model and code efficiency;
- Easy workload sharing in massive-parallel architectures.
- Is currently being employed...
  1. in ITER predictive analysis;
  2. in JET D-T stability studies [Ferreira EPS/IAEA 2015];
  3. in fast-ion experiment analysis on ASDEX-U.

ITER baseline scenario $I_p = 15$MA.

1. Which are the most unstable Alfvén Eigenmodes (AEs)?
2. Are stability properties sensitive to small perturbations?

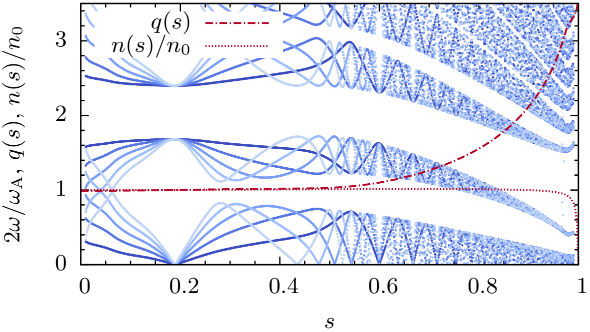


Plasma species temperature and density distributions [Polevoi 2002].

1:1 DT mix;
$B_0 = 5.3$ T;
$q_0 \approx 0.987$;

- Fusion-born α's mostly confined in the core ($s \lesssim 0.5$);
- No fast particles from auxiliary heating systems are considered.
- Peaked temperature profiles and flat density distribution.

Ideal Alfvén continuum structure.



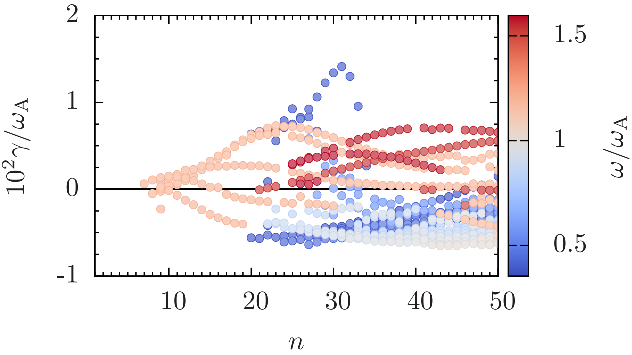
Alfvén continuum ($n = 10, \ldots, 50$, from dark to light hues), normalized density, and safety factor.

- Flat density up to the edge closes the frequency gaps;
- AEs extending towards the edge interact with the continuum;
- Flat $q(s)$ in the core promotes highly localized AEs;

How to scan the $(\omega, \mathbf{k})$-space.

- Sample the range $0 \leq \omega/\omega_A \leq 2$ in small steps ($\sim 10^{-5}$);
- Scan the range $1 \leq n \leq 50$, so that $k_\perp \rho_\alpha \approx (nq\rho_\alpha)/(ar) \lesssim 1$;

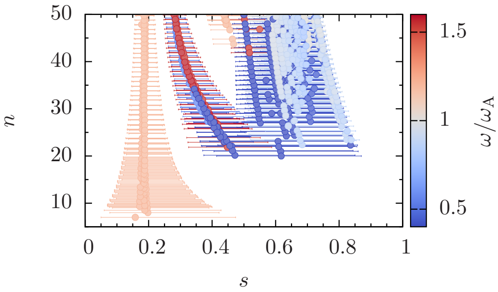
Stability results: γ/ω_A distribution by n and ω/ω_A.



Net γ/ω_A versus n for $\sim$ 700 AEs found in three frequency gaps: TAEs ($\omega/\omega_A \sim 0.5$), EAEs ($\omega/\omega_A \sim 1$), and NAEs ($\omega/\omega_A \sim 1.5$). Each AE is colored by its frequency. [Rodrigues 2015]

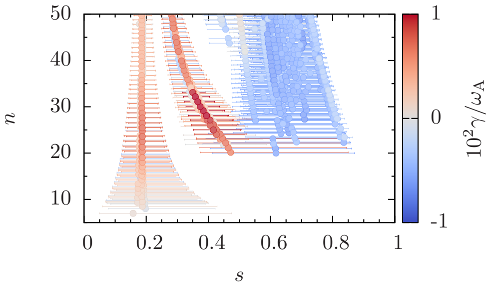
- Largest $\gamma/\omega_A = 1.5\%$ corresponds to a $n = 31$ TAE;
- EAEs and NAEs growth rates are in the range $\gamma/\omega_A \lesssim 0.7\%$.

Stability results: AEs radial location and width.



AEs radial localization (circles) and width (horizontal bars) distribution by toroidal number *n*. Each AE is colored by its normalized frequency (top) and growth rate (bottom). [Rodrigues 2015]

- Short-width unstable TAEs at $0.35 \lesssim s \lesssim 0.45$;
- Unstable EAEs at $s \approx 0.2$;
- Broad-width TAEs are stable;

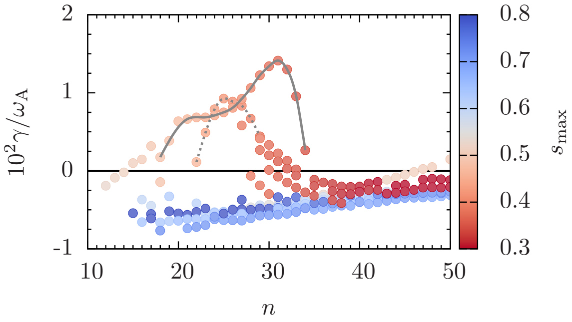


Partial summary II:

For the ITER baseline scenario considered:

- Core-localized, short-width TAEs ($10 \lesssim n \lesssim 30$) are the most unstable AE found;
- Normalized growth rates are of the order $\gamma/\omega_A \approx 1.5\%$;
- Broad-width AEs lie on the outer half of the plasma and most interact with the continuum;
- Consequences to α-particle transport are currently under investigation [Scheneller arXiv:1509.04010, Fitzgerald IAEA 2015].

The reference case ($I_{\mathrm{ref}} = 15$ MA and $q_0 = 0.987$).



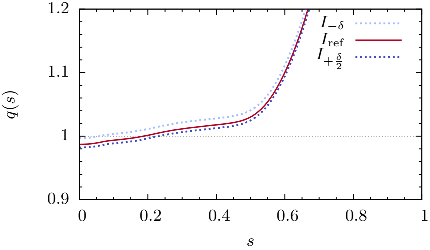
Net γ/ω_A distribution by toroidal mode number n for TAEs only;
Each mode is colored by the radial location of its maximum amplitude.
[Rodrigues 2015]

- Are stability properties sensitive to small changes of the background magnetic field?
- Which are their effects on $\frac{\gamma}{\omega_A}$ and n of the most unstable AEs?

Modified safety-factor profiles.

Reference case:

- Plasma current $I_{\text{ref}} = 15$ MA;
- On-axis safety factor $q_{\text{ref}} = 0.987$.



Safety-factor profiles for three values of I_{p}.

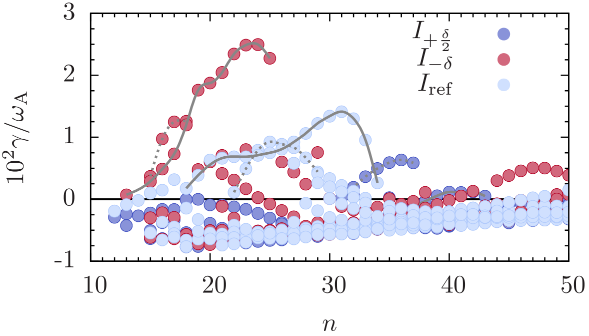
Modified magnetic equilibria:

- Keep the same equilibrium profiles $p'(\psi)$ and $f(\psi)f'(\psi)$;
- Change I_{p} from I_{ref} by δ and $\delta/2$, with $\delta = 0.16$ MA;

Effects of small plasma-current variations:

- On-axis value q_0 changes only slightly by circa 1% and 0.5%;
- Slope in the plasma core ($s \lesssim 0.5$) is kept unchanged.

Small variations of I_p: effects on AE stability.



Linear growth rate versus n for the three I_p values I_{ref}, $I_{+\frac{\delta}{2}}$, and $I_{-\delta}$.

Small variations ($\sim 1\%$) in I_p or q_0 cause **large changes** in n ($\sim 20\%$) and γ/ω_A ($\sim 50\%$) of the most unstable AEs:

- Lower I_p (higher q_0) raises γ/ω_A and reduces n;
- Higher I_p (lower q_0) reduces γ/ω_A and raises n;
- Most-unstable AEs are still even LSTAEs.

Understanding the sensitivity to small changes.

$$\begin{cases} q(s) = q_0 + q_0' s & \Leftarrow \text{low-shear region,} \\ q = 1 + 1/(2n) & \Leftarrow \text{ LSTAEs with } m = n, \\ k_\perp \Delta_\mathrm{orb} = \left(\frac{nq}{as}\right)\left(\frac{aq}{\varepsilon\tilde{\Omega}}\right) \sim 1 & \Leftarrow \text{drive-efficiency condition.} \end{cases}$$

The difference to a reference case (q_ref and n_ref) is then

$$\left(1+\frac{2\zeta-1}{4n_\mathrm{ref}\,n}\right)(n-n_\mathrm{ref}) = -\zeta(q_0-q_\mathrm{ref}), \quad \text{with} \quad \zeta = \frac{q}{q_0'}\frac{a}{\Delta_\mathrm{orb}} = \frac{\varepsilon\tilde{\Omega}}{q_0'}.$$

ITER parameters and consequences:

$q_0' \approx 0.07$, $\varepsilon = a/R_0 \approx 0.3$, $\tilde{\Omega} = \Omega_\alpha/\omega_\mathrm{A} \approx 230 \quad \Rightarrow \quad \zeta \approx 10^3$.

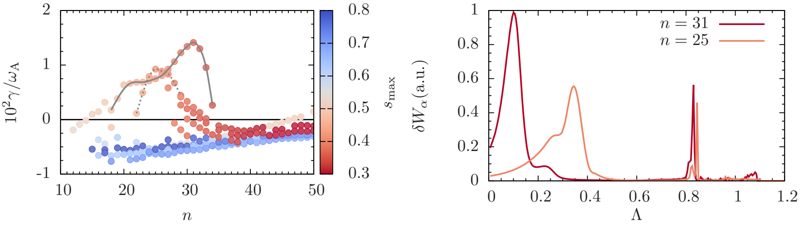
- **Large** ζ causes sensitivity to **small** changes $q_0 - q_\mathrm{ref}$;
- Raising q_0 above q_ref makes n drop below n_ref and vice-versa;

Partial summary III:

The stability of ITER baseline scenario was found to be highly sensitive to small changes in q_0 (or I_p);

- Cause large changes on n and γ/ω_A of the most unstable AEs;
- General feature, results from the large value $\zeta = \varepsilon\tilde{\Omega}/q_0'$;
- Further simulations (e.g., for α-particle transport) need to take such sensitivity into account when scenarios with low magnetic shear are being considered.

How α-particles interact with dominant AEs.



Resonant energy transfer δW_α by Λ value (right) for the two most unstable modes ($n = 25$ and $n = 31$) in each of the two TAE families (left).

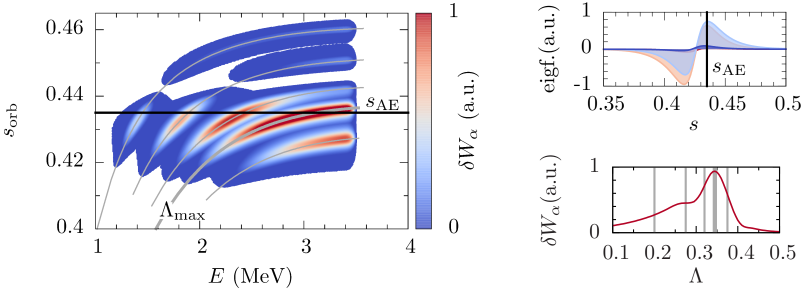
$$\Lambda \equiv \frac{\mu B_0}{E} = \frac{B_0}{B}\frac{E_\perp}{E}.$$

- Strongly passing particles, with small but finite Λ values (trapped particle $\Rightarrow \Lambda \gtrsim 1 - \varepsilon \approx 0.7$);
- Energy transfer is most efficient at $\Lambda_{max} = 0.1$ and 0.35 for the AEs with $n = 31$ and 25, respectively;

Why are some resonant particles more efficient than others?

Wave-particle resonance and AEs radial location.

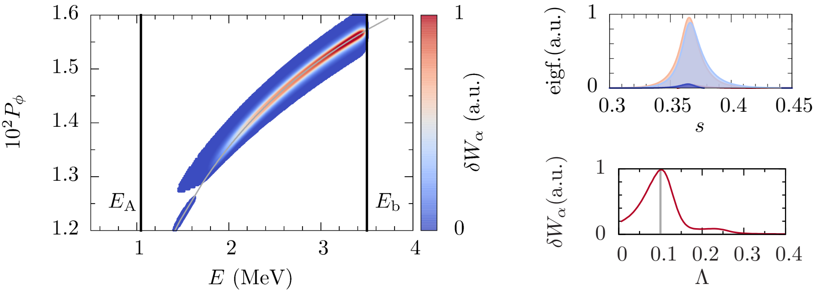
For an AE with given $(\omega, \mathbf{k})$: $\omega = \langle \mathbf{k} \cdot \mathbf{v} \rangle \Rightarrow \Gamma(s_{\mathrm{orb}}, E, \Lambda) = 0$.



Energy transfer δW_α from orbits drifting around the surface s_{orb}, with energy E and for five distinct Λ values. Resonance lines $\Gamma(s_{\mathrm{orb}}, E, \Lambda) = 0$ are in gray and the AE ($n = 25$) rational magnetic surface s_{AE} [such that $n\,q(s_{\mathrm{AE}}) = n + 1/2$] is in black.

- Distinct Λ select different resonance lines $\Gamma(s_{\mathrm{orb}}, E, \Lambda) = 0$;
- Along each line, the most efficient orbit drifts around s_{res};
- Λ_{max} corresponds to orbits able to drift around s_{res} with the maximum available energy: $E_{\mathrm{b}} = 3.5$ MeV.

Wave-particle resonance at $v_\parallel > v_A$.



Energy transfer δW_α from orbits with energy E, normalized toroidal momentum P_ϕ, and Λ_{max} for the most unstable AE in the set ($n = 31$).

The most efficient orbits at Λ_{max}...

- are close to the maximum available energy E_b;
- are well above the on-axis Alfvén energy $E_A = \frac{1}{2} m v_A^2$;
- have a ratio $(v_\parallel / v_A) \sim \sqrt{(E/E_A)(1 - \Lambda_{max})} \approx 1.8 > 1$.

Resonance condition for co-passing particles.

$$\omega + (l - m)\omega_\theta + n\omega_\phi = 0, \quad \text{with} \quad l = \pm 1, \pm 2, \ldots$$

Using usual estimates:

[Heidbrink 2007]

- Assume the drift velocity to be small ($v_\perp << v_\|$);
- The toroidal and poloidal circulation frequencies are $\omega_\phi \approx v_\| / R_0$ and $\omega_\theta \approx v_\| / (qR_0)$, respectively;
- The AE's frequency is $\omega \approx \frac{\nu}{2q} \frac{v_\mathrm{A}}{R_0}$;
- AEs couple at the resonant surface $q = (m + \frac{\nu}{2})/n$, where $\nu = 1$ (TAEs), 2 (EAEs), $\ldots$ is the frequency gap index;

Consequences:

$$\frac{v_\|}{v_\mathrm{A}} = \frac{\nu}{2l - \nu} \leqslant 1 \quad \Leftarrow \quad \text{if} \quad \nu = 1 \quad \text{(TAEs)}$$

Drift-velocity effects on wave-particle resonance.

$$\omega + \langle \mathbf{k} \cdot \mathbf{v} \rangle = 0$$

- Separate $\mathbf{v} = v_\parallel \mathbf{b} + \mathbf{v}_\perp$ into parallel and drift components;
- Use the shear-Alfvén wave dispersion relation $\omega^2 = k_\parallel^2 v_A^2$;

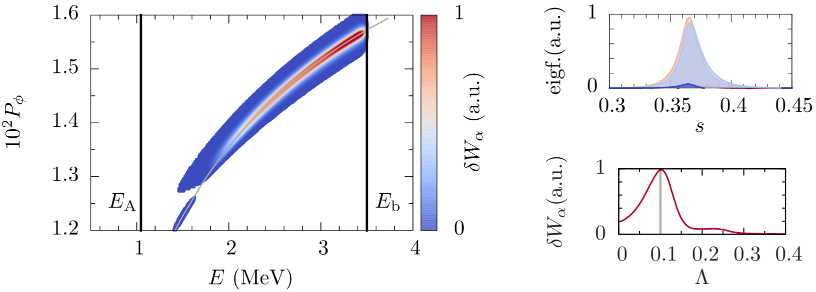
$$1 - \left\langle \frac{v_\parallel}{v_A} \right\rangle + \left\langle \frac{\mathbf{k}_\perp \cdot \mathbf{v}_\perp}{\omega} \right\rangle = 0$$

When are drift-velocity effects important?

- Simple cylindrical equilibrium: $k_\perp \sim \frac{nq}{ar}$ and $v_\perp \sim \frac{v_\parallel^2}{R_0 \Omega} \frac{\varepsilon r}{q^2}$;
- Therefore, $\left\langle \frac{\mathbf{k} \cdot \mathbf{v}_\perp}{\omega} \right\rangle \sim \left\langle \frac{2n}{\tilde{\Omega}} \left(\frac{v_\parallel}{v_A} \right)^2 \right\rangle \sim \frac{2n}{\tilde{\Omega}} \frac{E}{E_A} (1 - \Lambda)$;
- For small Λ and ITER values $E/E_A \approx 3.5$ and $\tilde{\Omega} = 230$:
  $10 \lesssim n \lesssim 50 \quad \Rightarrow \quad 0.3 \lesssim \left\langle \frac{\mathbf{k}_\perp \cdot \mathbf{v}_\perp}{\omega} \right\rangle \lesssim 1.5.$

The drift term is important in the relevant *n* range for ITER.

Estimate of drift effects for the $n = 31$ TAE.

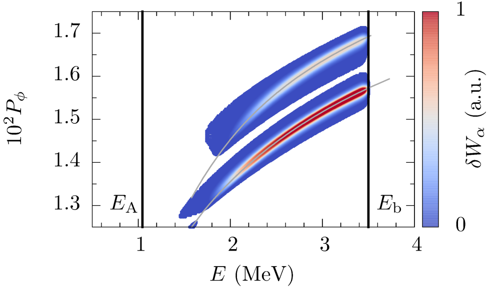


Energy transfer δW_α from orbits with energy E, normalized toroidal momentum P_ϕ, and Λ_{max} for the most unstable TAE in the set ($n = 31$).

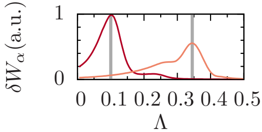
Using a simple cylindrical equilibrium approximation:

- $\langle v_{\|}/v_\mathrm{A} \rangle \sim \sqrt{(E/E_\mathrm{A})(1-\Lambda_{\mathrm{max}})} \approx 1.8$;
- $\langle \omega^{-1} \mathbf{k}_\perp \cdot \mathbf{v}_\perp \rangle \sim \frac{2n}{\tilde{\Omega}} \frac{E}{E_\mathrm{A}} \left(1 - \Lambda_{\mathrm{max}}\right) \approx 0.9$;
- Therefore, $1 - \langle v_{\|}/v_\mathrm{A} \rangle + \langle \omega^{-1} \mathbf{k}_\perp \cdot \mathbf{v}_\perp \rangle \approx 0$ within 10%.

Distinct efficiencies of energy transfer from E_b.



Energy transfer δW_α from orbits with E, P_ϕ, and two Λ_{max} values for the $n = 25$ and $n = 31$ TAEs.



- Drift effects are important for all n of interest;
- Most unstable AEs are able to access E_b via a resonance line;
- The efficiency of the energy transfer from orbits at E_b changes with the AEs characteristics (n, ω, Λ_{max}, etc.).

Can drift corrections to the resonance relation provide clues about optimal AE parameters?

A quadratic form of the resonance relation.

$$\omega + (l - m)\omega_\theta + n\omega_\phi = 0, \quad \text{with} \quad l = \pm 1, \pm 2, \ldots$$

- Let $l = -1$ and split $\omega_\phi = \frac{v_\parallel}{R_0} + \omega_\phi^1$ and $\omega_\theta = \frac{v_\parallel}{qR_0} + \omega_\theta^1$;

$$\omega + n\left\langle \omega_\phi^1 - q\omega_\theta^1 \right\rangle - \left\langle \frac{1}{2q}\frac{v_\parallel}{R_0} \right\rangle = 0$$

- First-order drift terms are

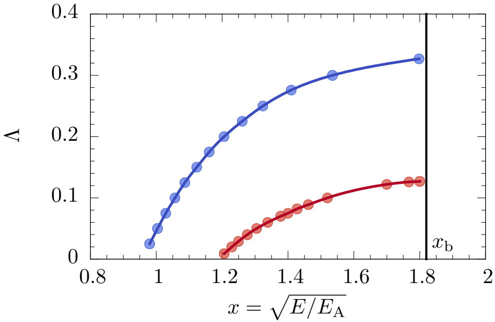
$$\omega_\phi^1, \omega_\theta^1 \propto \frac{v_\parallel^2}{R_0^2\Omega} = \frac{\omega_A}{\tilde{\Omega}}\frac{E}{E_A}\left(\frac{B_0}{B} - \Lambda\right);$$

- Define $x^2 \equiv E/E_A$ and let $f_\parallel(q,\Lambda)$ and $f_\perp(q,\Lambda)$ be unknown functions resulting from the averages of ω_ϕ^1, ω_θ^1, and $v_\parallel/q$;

$$\Gamma(x,\Lambda; n,\tilde{\omega},q) \equiv n\,\tilde{\Omega}^{-1}\, f_\perp(q,\Lambda)\, x^2 - f_\parallel(q,\Lambda)\, x + \tilde{\omega} = 0$$

For an AE with given n, $\tilde{\omega}$, and $q = (m+1/2)/n$, the condition $\Gamma(x,\Lambda; n,\tilde{\omega},q) = 0$ defines a resonant line in the (x,Λ) plane.

Properties of the quadratic resonance relation.



Points along resonance lines $\Gamma(x, \Lambda)$ as computed by `CASTOR-K` for the $n = 25$ (blue) and $n = 31$ (red) TAEs. The black line stands for the birth-energy limit x_b.

- Lines intersect $x_\mathrm{b} = \sqrt{E_\mathrm{b}/E_\mathrm{A}}$ at the value Λ_max;
- Only lines with $0 \leq \Lambda_\mathrm{max} \leq 1 + \varepsilon$ can access orbits with E_b;
- Solving $\Gamma(x_\mathrm{b}, \Lambda_\mathrm{max}) = 0$ demands $f_\parallel$ and $f_\perp$ to be known;
- Most efficient energy transfer takes place when $(x_\mathrm{b}, \Lambda_\mathrm{max})$ is a local extremum of the resonance line $\Gamma(x, \Lambda) = 0$.

Drift-velocity condition for energy-transfer efficiency.

For general quadratic equations:

- At local extrema, solutions of $ax^2 - bx + c = 0$ are degenerate;

$$x = b/(2a) \quad \text{and} \quad b^2 - 4ac = 0$$

- Therefore, $ax^2 = c$ regardless of the particular a and b values;
- At local extrema, the drift term in the resonance relation is

$$\langle \mathbf{k}_\perp \cdot \mathbf{v}_\perp \rangle = n\,\Omega^{-1}\, f_\perp(q,\Lambda)\, x^2 = \omega$$

regardless of the unknown functions $f_\perp(q,\Lambda)$ and $f_{||}(q,\Lambda)$.

Condition of efficient energy transfer at E_b:

$$\left\langle \frac{\mathbf{k}_\perp \cdot \mathbf{v}_\perp}{\omega} \right\rangle \Bigg|_{(E_\mathrm{b},\Lambda_\mathrm{max})} = 1$$

Partial summary IV:

The properties of wave-particle resonant interaction in ITER baseline scenario were addressed;

- The energy-transfer efficiency of resonant orbits was discussed;
- Drift-velocity effects in the resonance condition were found to be important;

Conclusions:

1. An approach was developed to handle routine stability assessments and sensitivity analysis in burning plasmas;
   - Hybrid model and code efficiency;
   - Easy workload sharing in massive-parallel architectures.
2. For the ITER baseline scenario considered:
   - Core-localized TAEs ($10 \lesssim n \lesssim 30$) are the most unstable;
   - Normalized growth rates are of the order $\gamma/\omega_\mathrm{A} \approx 1.5\%$;
3. The stability of ITER baseline scenario was found to be highly sensitive to small changes in q_0 (or I_p);
   - Cause large changes on n and γ/ω_A of the most unstable AEs;
   - General feature, results from large value $\zeta = \varepsilon\tilde{\Omega}/q_0'$;
4. The properties of wave-particle resonant interaction in ITER baseline scenario were addressed;
   - The energy-transfer efficiency of resonant orbits was discussed;
   - Drift-velocity effects in the resonance condition were found to be important;

Backup slides ahead.

Hybrid model.

Distribution functions:

- Thermal species (DT ions, electrons, He ash) are Maxwellian;
- Fusion-born αs are isotropic and follow the slowing-down distribution

$$f_{\mathrm{sd}}(E) = \frac{1}{E^{3/2} + E_{\mathrm{c}}^{3/2}}\,\mathrm{erfc}\left(\frac{E - E_0}{\Delta_{\mathrm{E}}}\right)$$

Population separation:

- Bulk plasma collectively described by ideal-MHD theory; (p_{MHD}, n_{MHD}, ϱ, $\mathbf{v}$, $\mathbf{J}$)
- Evolution of the non-Maxwellian αs described by a drift-kinetic equation;

$$\omega / \Omega_\alpha \sim k_\perp \rho_\alpha \ll 1$$

How the MHD and the kinetic models are linked.

1. Fusion αs are a very diluted population, with $n_\alpha / n_{\mathrm{MHD}} \ll 1$;
2. Fluid and kinetic models interact via the pressure tensor only;

Equilibrium quantities:

- The overall pressure is the sum $p = p_{\mathrm{MHD}} + p_\alpha$;
- $p_\alpha / p_{\mathrm{MHD}} \sim (n_\alpha / n_{\mathrm{MHD}})(E_\alpha / E_{\mathrm{MHD}})$ is not necessarily small;
- p_α must be accounted for in the magnetic equilibrium;
- The αs contribution to the overall current is negligible;

$$J_\alpha / J \sim Z_\alpha (n_\alpha / n_{\mathrm{MHD}})(m_\mathrm{e} / m_\alpha)^{1/2}(E_\alpha / E_{\mathrm{MHD}})^{1/2} \ll 1$$

First-order perturbations:

- The pressure tensor splits as $\delta \mathsf{p} = \delta p_{\mathrm{MHD}} + \delta \mathsf{p}_\alpha$;
- the energy principle becomes

$$\omega^2 W_\mathrm{K} = \delta W_{\mathrm{MHD}} + \delta W_\alpha, \qquad \delta W_\alpha / \delta W_{\mathrm{MHD}} \ll 1.$$

How each perturbation is computed.

$$\left(\omega_{\mathrm{MHD}} + \delta\omega\right)^2 W_{\mathrm{K}} = \delta W_{\mathrm{MHD}} + \delta W_\alpha, \qquad \frac{\delta\omega}{\omega_{\mathrm{MHD}}} \sim \frac{\delta W_\alpha}{\delta W_{\mathrm{MHD}}} \ll 1.$$

1. $\delta\varrho$, δp, $\delta\mathbf{v}$, and $\delta\mathbf{A}$ are found from $\omega_{\mathrm{MHD}}^2 W_{\mathrm{K}} = \delta W_{\mathrm{MHD}}$;
2. The αs response δf_α to the MHD perturbation is

$$\delta f_\alpha = -\mathrm{i}\left(\omega - n\omega_*\right)\frac{\partial f_\alpha}{\partial E}\int d\tau\, \delta L(\tau),$$

$$\delta L = eZ_\alpha\left(\delta\mathbf{A}\cdot\dot{\mathbf{X}} - \delta\Phi\right) - \mu\,\delta B, \quad \omega_* \equiv \left(\partial f_\alpha/\partial P_\phi\right)/\left(\partial f_\alpha/\partial E\right);$$

3. The energy exchanged is the phase-space integral

$$\delta W_\alpha = -\frac{1}{2}\int d^3x\, d^3v\, \delta L^\dagger \delta f_\alpha;$$

[Porcelli 1994]

4. The frequency correction $\delta\omega$ due to the α interaction is:

$$\frac{\delta\omega}{\omega_{\mathrm{MHD}}} = \frac{\delta W_\alpha}{2\omega_{\mathrm{MHD}}^2 W_{\mathrm{K}}}.$$