Simulation of reflectometry using Finite-Difference Time-Domain Codes

Filipe da Silva

[1] Stéphane Heuraux

[2] Tiago Ribeiro

[3] Bruno Després

[3] Martin Campos Pinto

Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico Lisbon, Portugal

[1] Institut Jean Lamour, UMR 7198 CNRS-University Lorraine, Vandoeuvre, France

[2] Max-Planck-Institut für Plasmaphysik, 85748 Garching, Germany

[3] Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

Lisboa, 12 Dezembro, 2014, GTM Seminar

Roadmap...

- Brief contextualization of reflectometry
- The requested sort of tutorial on FDTD
- Maxwell FDTD
- **REFMUL**: O-mode code
  - *Example of a forward scattering simulation*
- **REFMULX**: X-mode code
  - *Example of a synthetic diagnostic with GEMR input*
- **REFMULF**: Full polarization code
  - *Example of the first tests*
- First new areas of application

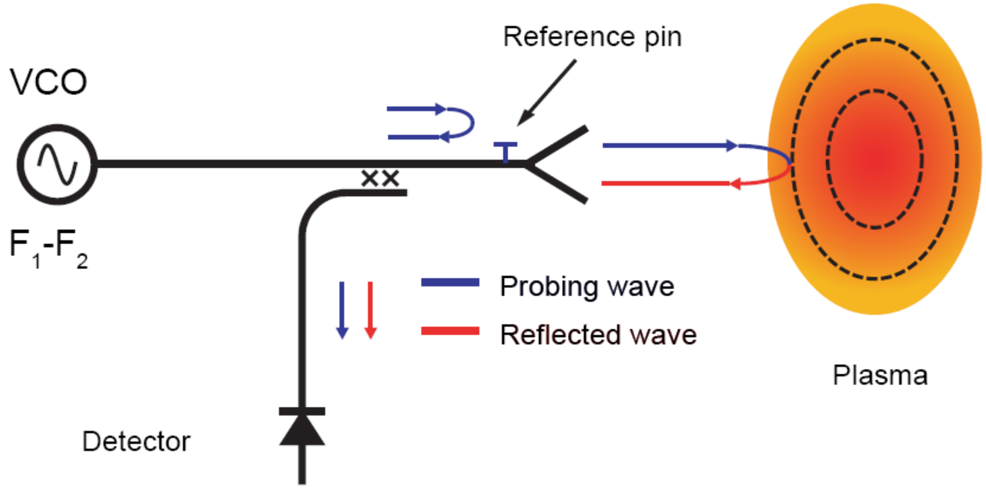
Reflectometry in a nutshell

- Signal sent to the plasma

$$s_{src}(t) = A_{src}\cos(\omega t)$$

- It propagates being reflected at the cutoff position r_c

$$s_{rcv}(t) = A_{ant}\cos\left(\omega t + \varphi\right)$$



$$\varphi = \varphi_0 + \varphi_\mu + \varphi_p$$

- The reflected wave shows a phase shift φ due to the microwave circuit φ_μ, propagation in a vacuum φ_0 and in the **plasma φ_p**.

$$\varphi_p = \frac{4\pi f}{c}\int_0^{r_c} N(r)\mathrm{d}r - \varphi_k$$

- The phase φ_p reflects the propagation of the wave along a path described by a refraction index *N(r)* and contains information about **electronic density n_e**

- O-mode $\varphi_p = \varphi_p[f, N_O(n_e)]$
- X-modo $\varphi_p = \varphi_p[f, N_X(n_e, B_0)]$

Altar-Appleton (Appleton-Hartree) equation

- Solving the dispersion equation in N^2

$$N^2 = 1 - \frac{X(1-X)}{1 - X - 1/2Y^2\sin^2\theta \pm \left[(1/2Y^2\sin^2\theta)^2 + (1-X)^2Y^2\cos^2\theta\right]^{1/2}}$$

- Of interest to reflectometry is the case with θ=π/2, which gives 2 expressions:

$$N^2 = 1 - X$$

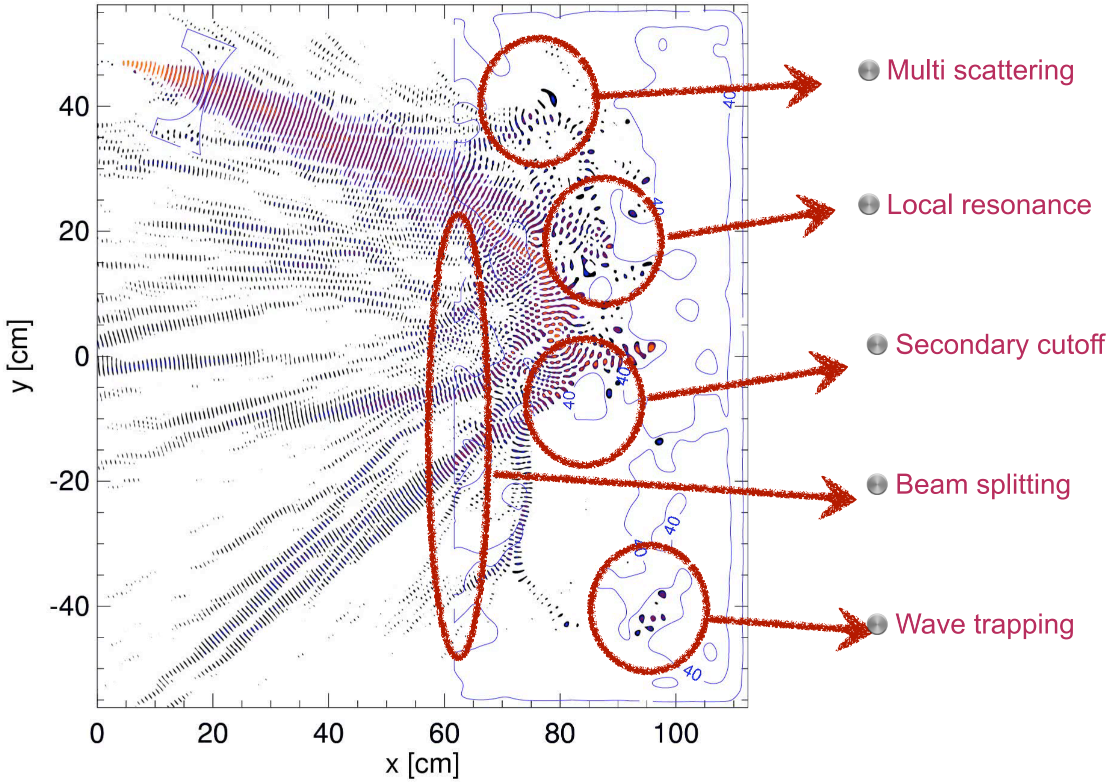
- **O-mode** - linear polarization with **$E//B_0$**

$$N^2 = 1 - \frac{X(1-X)}{1 - X - Y^2}$$

- **X-mode**- elliptic polarization with **$E \perp B_0$**

Propagation on plasma far from trivial

Plasma is in reality an extremely complex, non-homogeneous, non-stationary, anisotropic, where waves suffer the effects of turbulence, MHD, Doppler shifts, absorption, diffusion, mode conversion. It requires a numerical full-wave treatment based on a simplified model which retains the fundamental physics.



Several CEM techniques have been used to tackle this problem such as screen-phase, ray-tracing, Helmholtz, TML, FEM, paraxial approximation and wave-equation solver but all present limitations.

The technique most up to the job is FDTD.

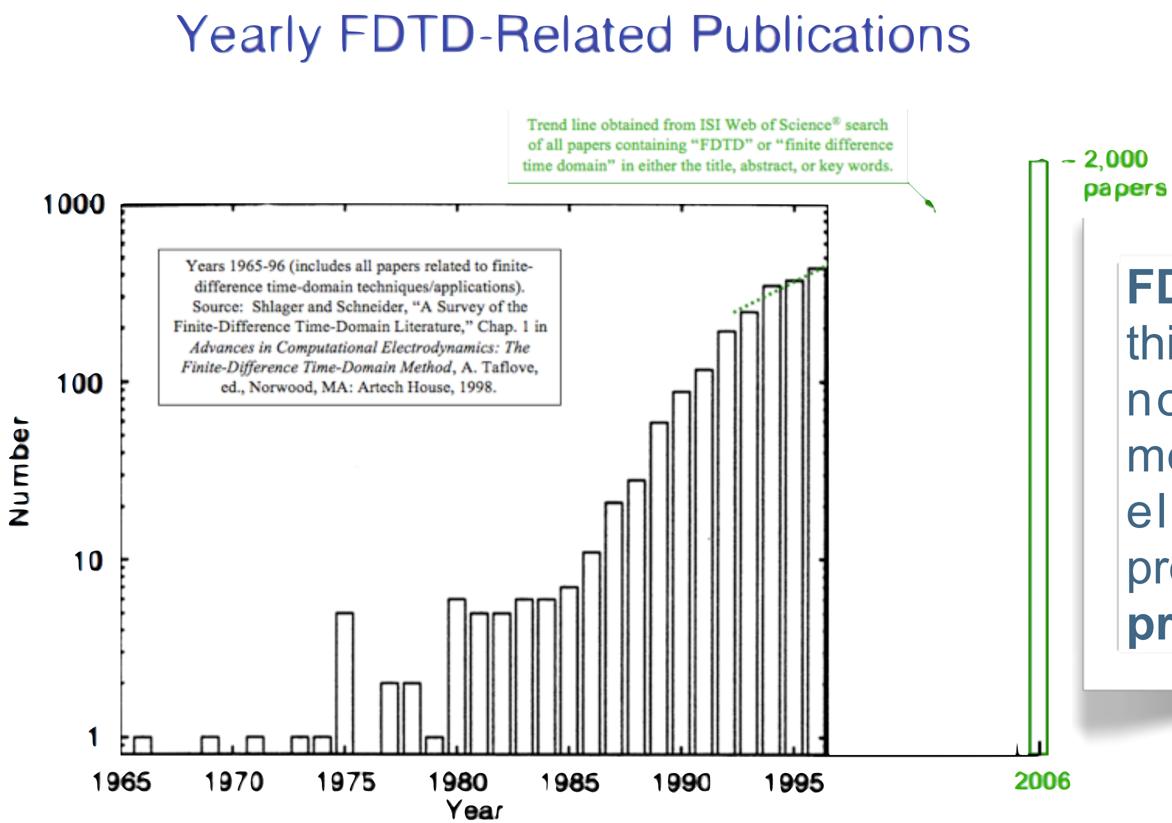
Finite-Difference Time-Domain

Finite-difference time-domain (FDTD) is a numerical analysis technique used in computational electrodynamics.

Finds approximate solutions to the associated system of differential equations.

As a time-domain method, covers a wide frequency range in a single run.

Treats nonlinear material properties in a natural way.



FDTD has become, in the past thirty years, one of the major, if not the principal numerical method to solve problems of electromagnetism, including problems in the area of **wave propagation**.

Propagating-wave solution

$$\frac{\partial^2 u}{\partial t^2} = \mathrm{c}^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x,t) = F(x+\mathrm{c}t) + G(x-\mathrm{c}t)$$

$$\frac{\partial u}{\partial t} = \underbrace{\frac{\mathrm{d}F(x+\mathrm{c}t)}{\mathrm{d}(x+\mathrm{c}t)}}_{F'} \cdot \underbrace{\frac{\partial(x+\mathrm{c}t)}{\partial t}}_{\mathrm{c}} + \underbrace{\frac{\mathrm{d}G(x-\mathrm{c}t)}{\mathrm{d}(x-\mathrm{c}t)}}_{G'} \cdot \underbrace{\frac{\partial(x-\mathrm{c}t)}{\partial t}}_{-\mathrm{c}}$$

$$\frac{\partial u}{\partial t} = \mathrm{c}F'(x+\mathrm{c}t) - \mathrm{c}G'(x-\mathrm{c}t)$$

$$\frac{\partial^2 u}{\partial t^2} = \mathrm{c}^2 F''(x+\mathrm{c}t) + \mathrm{c}^2 G''(x-\mathrm{c}t)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x+\mathrm{c}t) + G''(x-\mathrm{c}t)$$

$$\mathrm{c}^2 F''(x+\mathrm{c}t) + \mathrm{c}^2 G''(x-\mathrm{c}t) = \mathrm{c}^2 \left[F''(x+\mathrm{c}t) + G''(x-\mathrm{c}t)\right]$$

Finite differences

$$\frac{\partial^2 u}{\partial t^2} = \mathrm{c}^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x,t) = F(x+\mathrm{c}t) + G(x-\mathrm{c}t)$$

expanding u(x,t) about a space point x_i to $x_i+\Delta x$, keeping time t_n fixed:

$$u(x_i+\Delta x)\Big|_{t_n} = u\Big|_{x_i,t_n} + \Delta x \frac{\partial u}{\partial x}\Big|_{x_i,t_n} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_{x_i,t_n} + \frac{(\Delta x)^3}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{x_i,t_n} + \frac{(\Delta x)^4}{24}\frac{\partial^4 u}{\partial x^4}\Big|_{\xi_1,t_n}$$

expanding u(x,t) about a space point x_i to $x_i-\Delta x$, keeping time t_n fixed:

$$u(x_i-\Delta x)\Big|_{t_n} = u\Big|_{x_i,t_n} - \Delta x \frac{\partial u}{\partial x}\Big|_{x_i,t_n} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_{x_i,t_n} - \frac{(\Delta x)^3}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{x_i,t_n} + \frac{(\Delta x)^4}{24}\frac{\partial^4 u}{\partial x^4}\Big|_{\xi_2,t_n}$$

summing: $$u(x_i+\Delta x)\Big|_{t_n} + u(x_i-\Delta x)\Big|_{t_n} = 2u\Big|_{x_i,t_n} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_{x_i,t_n} + \frac{(\Delta x)^4}{12}\frac{\partial^4 u}{\partial x^4}\Big|_{\xi_3,t_n}$$

Second-order accurate central-difference approximation to the second partial space derivative of u

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x_i,t_n} = \left[\frac{u(x_i+\Delta x) - 2u(x_i) + u(x_i-\Delta x)}{(\Delta x)^2}\right]_{t_n} + O\left[(\Delta x)^2\right]$$

FD approximation of the scalar wave equation

$$u(x_i, t_n) = u(i\Delta x, n\Delta t) = u_i^n$$

space derivative:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i, t_n} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + O\Big[(\Delta x)^2\Big]$$

time derivative:

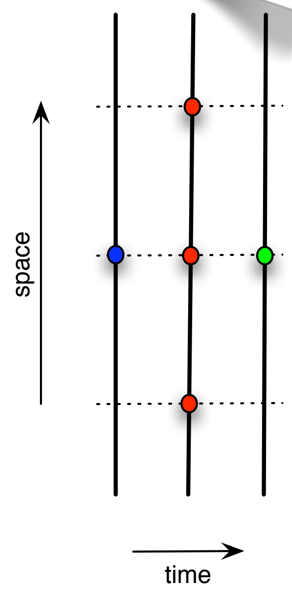
$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{x_i, t_n} = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} + O\Big[(\Delta t)^2\Big]$$

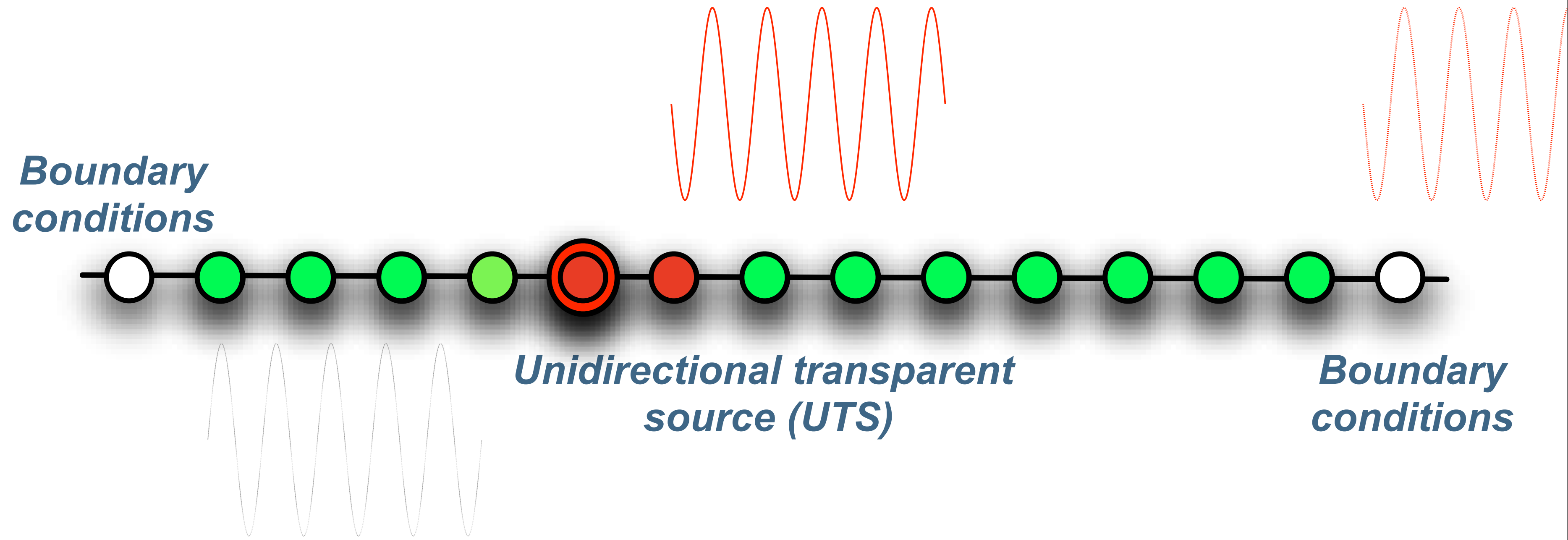
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} + O\Big[(\Delta t)^2\Big] = c^2 \left\{ \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + O\Big[(\Delta x)^2\Big] \right\}$$

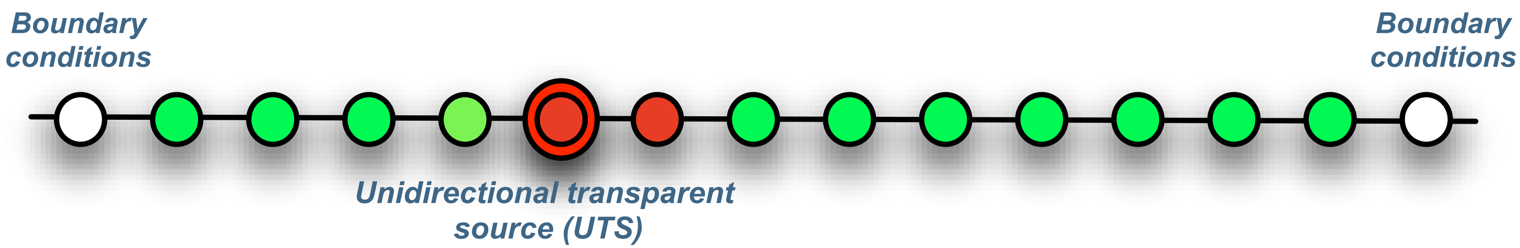
fully explicit 2nd-order accurate expression for u_i^{n+1}:

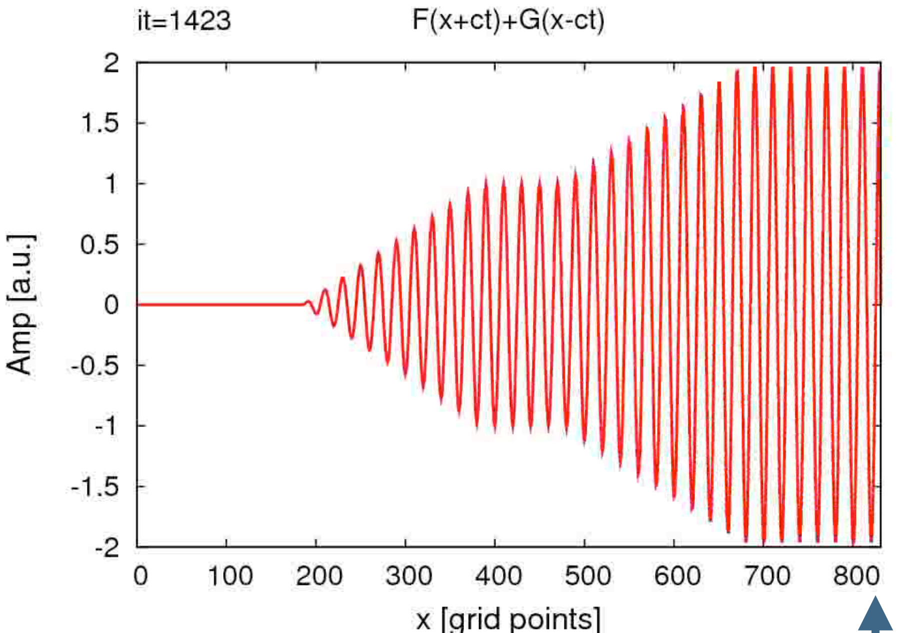
$$u_i^{n+1} \cong (c\Delta t)^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + 2u_i^n - u_i^{n-1}$$

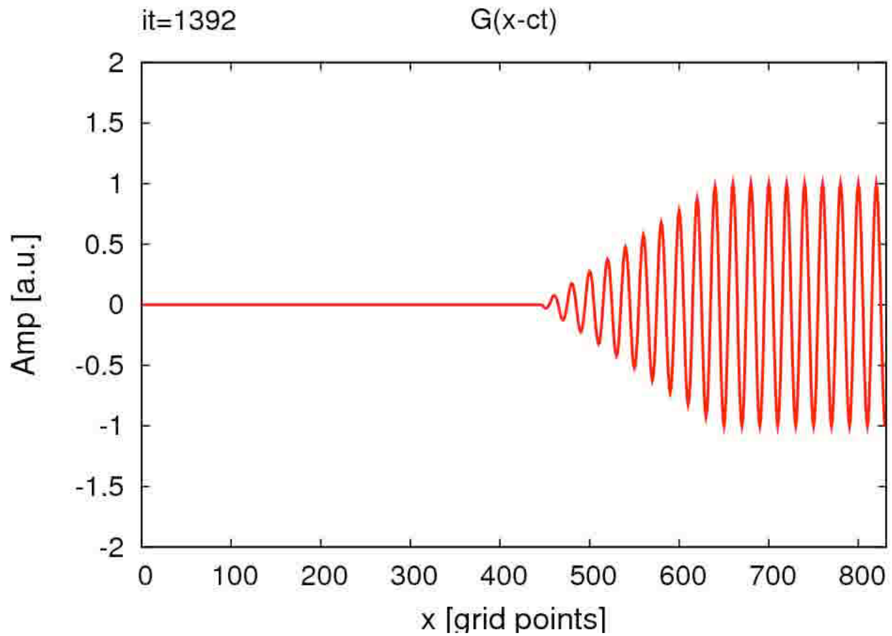



Boundary conditions
Unidirectional transparent source (UTS)
Boundary conditions

Example of a sinusoidal excitation



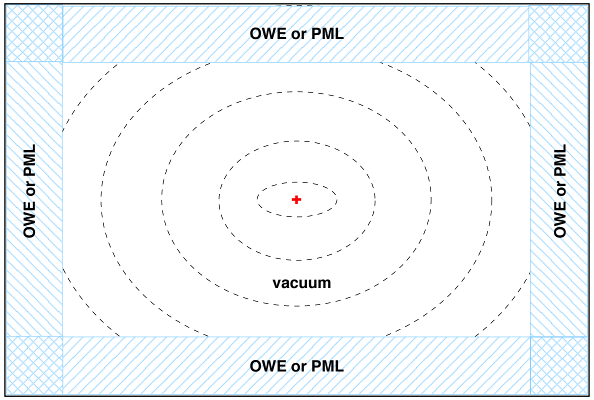




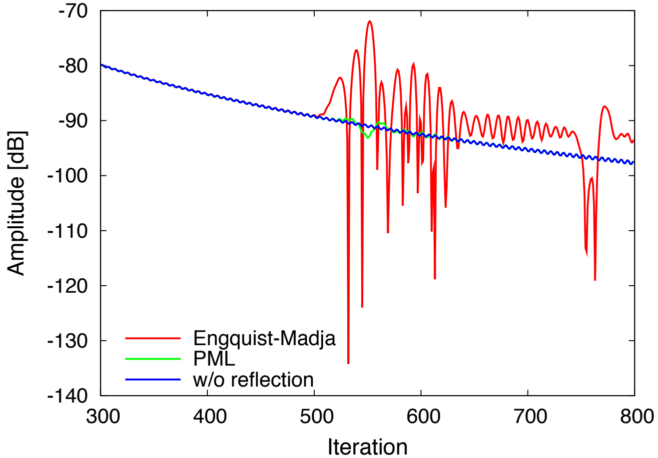
No boundary condition

Boundary conditions (2D)

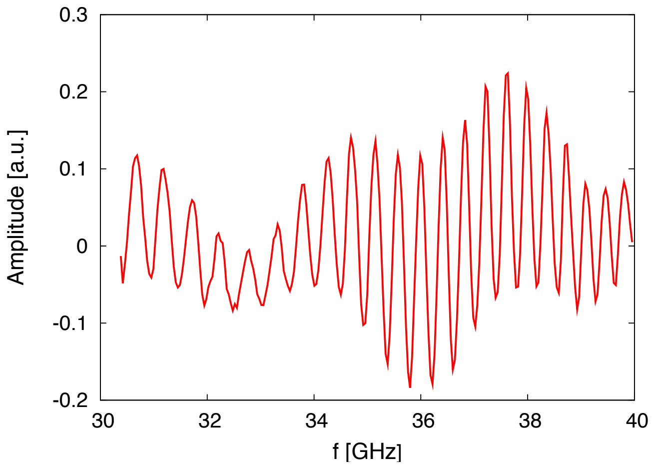
Calculation domain



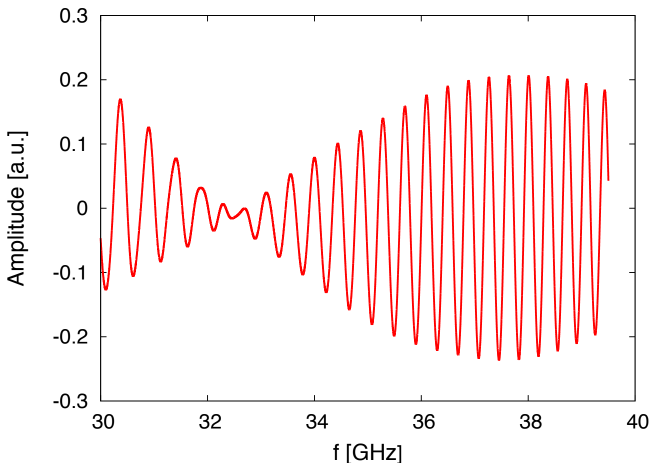
Response from boundary



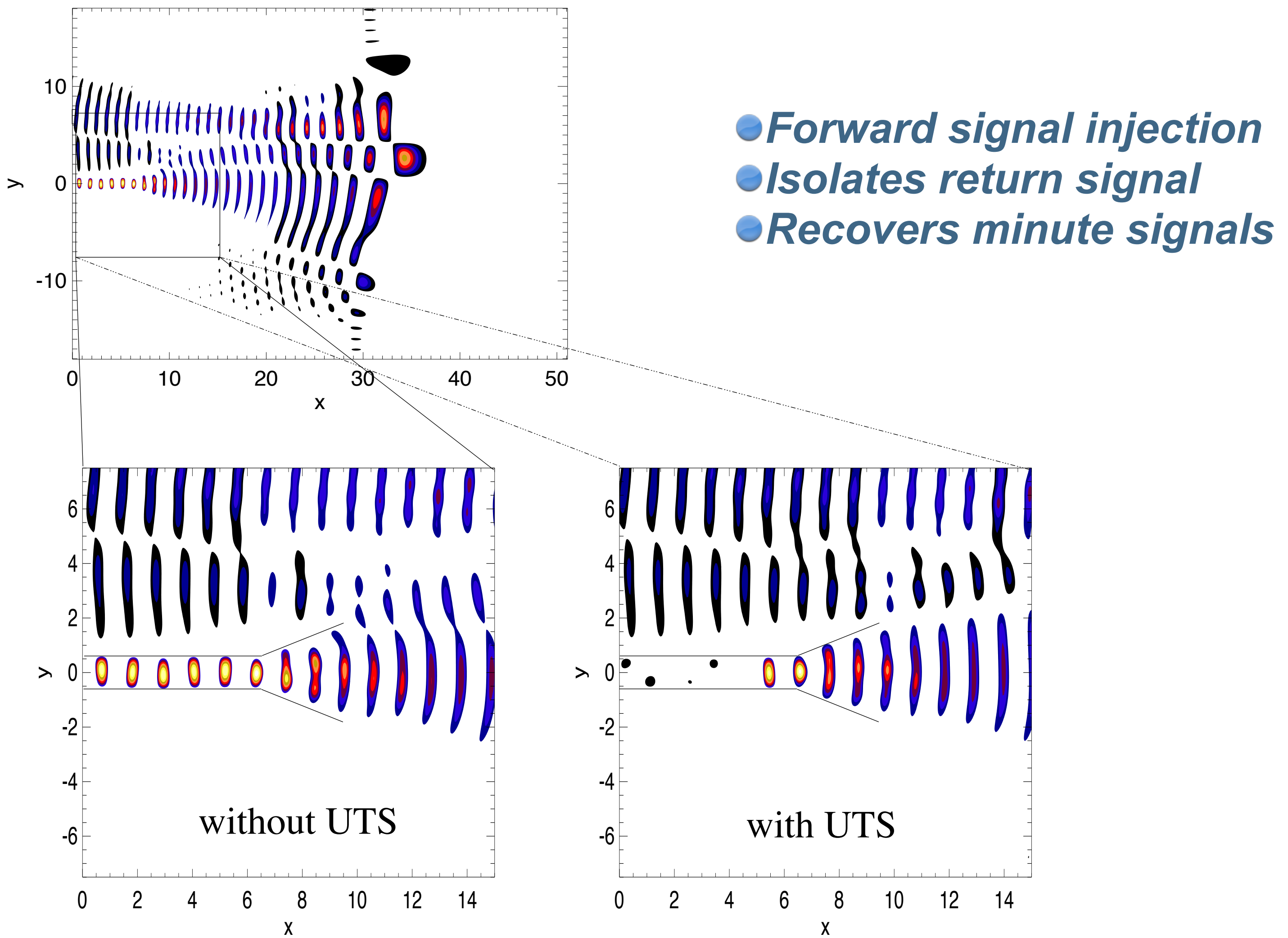
One-way equation



Perfectly matched layer



Unidirectional Transparent Source (2D)



- ***Forward signal injection***
- ***Isolates return signal***
- ***Recovers minute signals***

(O)UTS @ Backup slides

Some important concepts

Numerical dispersion

$$\tilde{k} = \frac{1}{\Delta x} \cos^{-1} \left\{ 1 + \left(\frac{\Delta x}{c \Delta t} \right)^2 \left[\cos(\omega \Delta t) - 1 \right] \right\}$$

Numerical stability

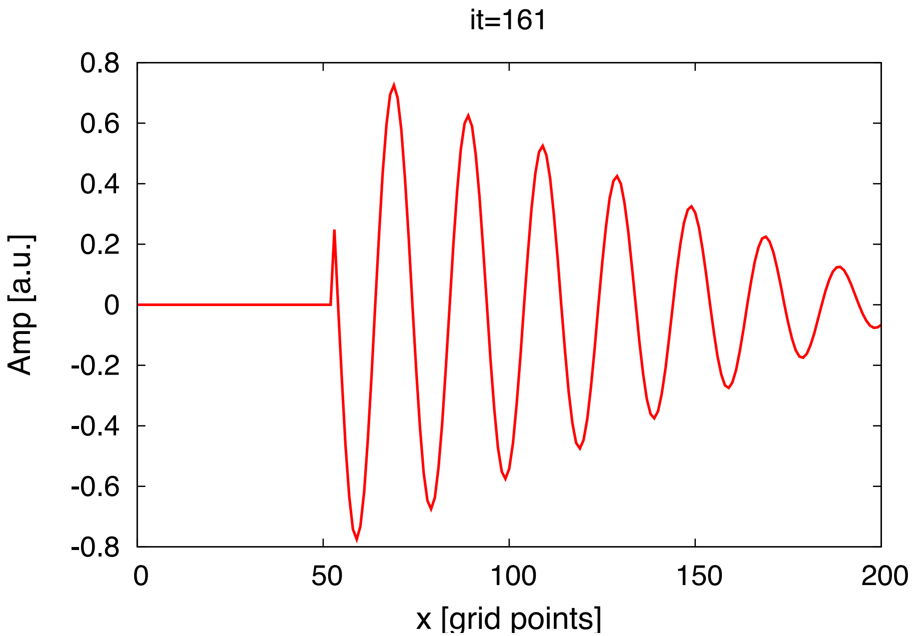
$$u_i^n = e^{j(\tilde{\omega}_{real} n \Delta t - \tilde{k} i \Delta x)}$$

$$u_i^n = \left(\frac{1}{-\xi - \sqrt{\xi^2 - 1}} \right)^{nth \text{ power}} e^{j\left[(\pi/\Delta t) n \Delta t - \tilde{k} i \Delta x\right]}$$

(Un)Stability example...

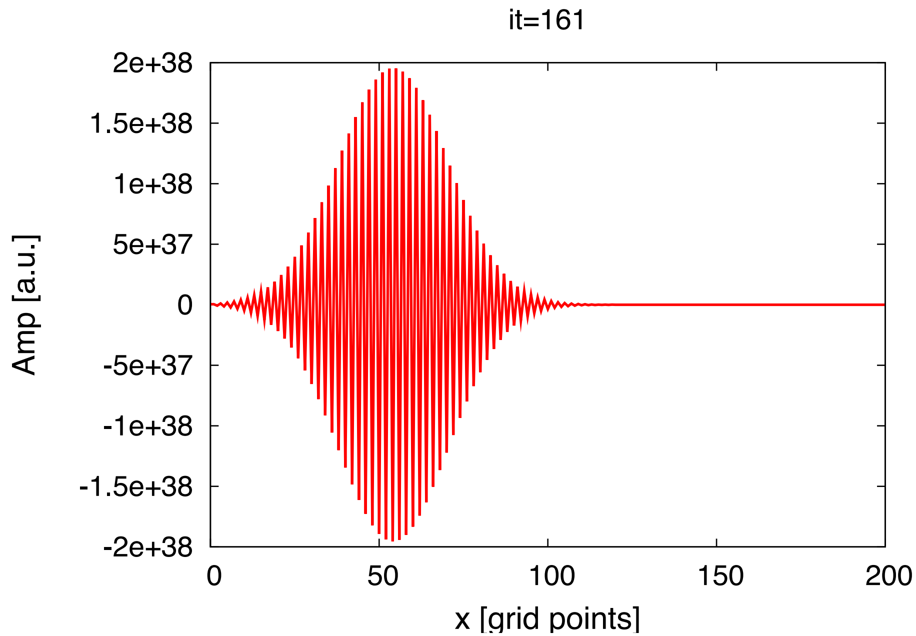
$$S = c\Delta t/\Delta x = 1.00$$

$$u_i^n = \mathrm{e}^{j(\tilde{\omega}_{real} n\Delta t - \tilde{k} i \Delta x)}$$

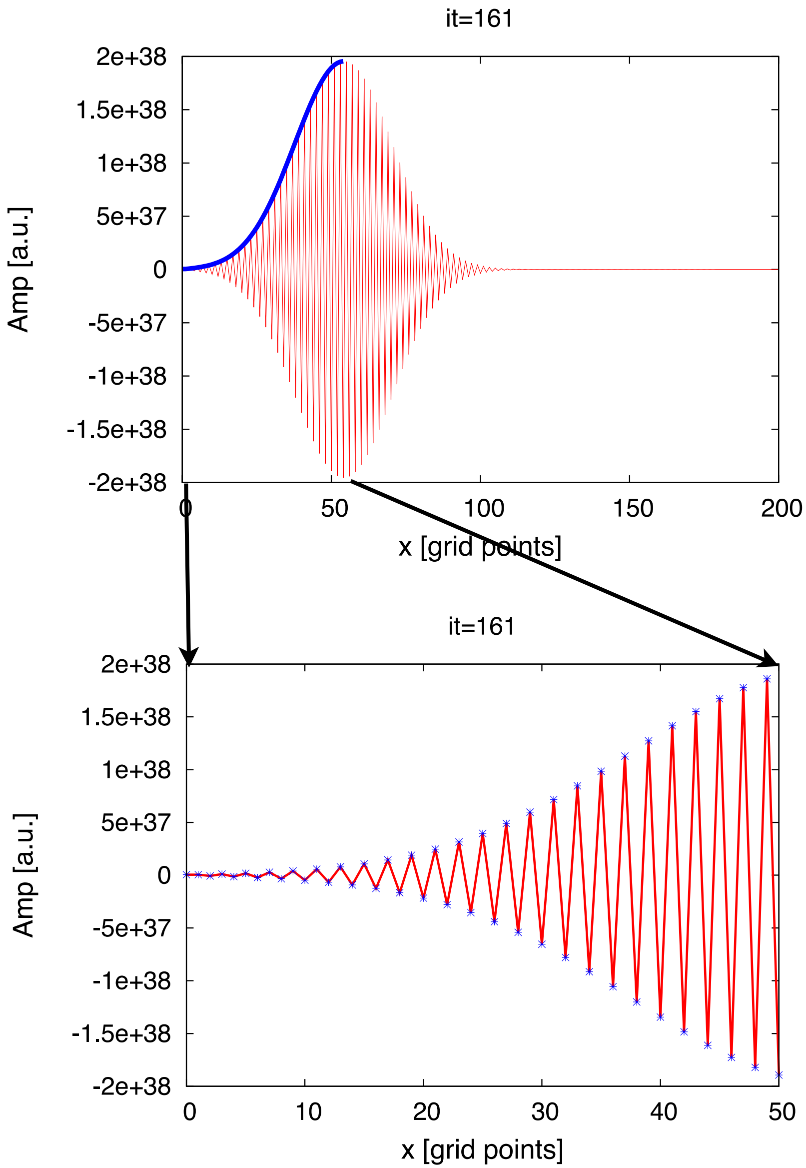


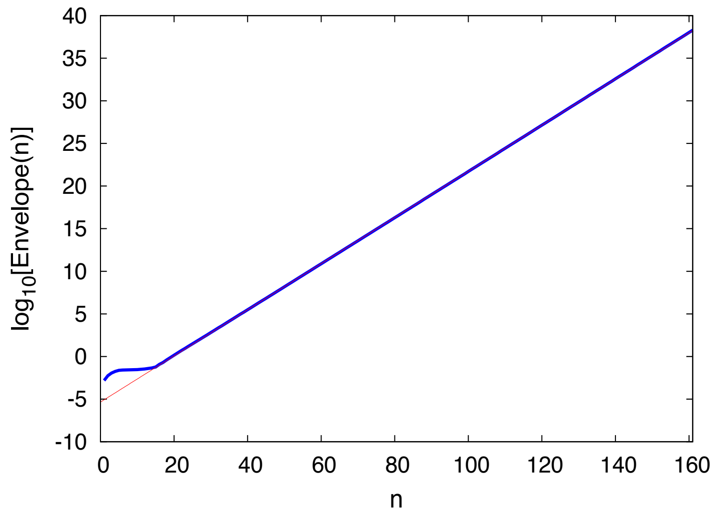
$$S = c\Delta t/\Delta x = 1.05$$

$$u_i^n = \left(\frac{1}{-\xi - \sqrt{\xi^2 - 1}} \right)^{nth\,\text{power}} \mathrm{e}^{j\left[(\pi/\Delta t) n\Delta t - \tilde{k} i \Delta x\right]}$$



Looking into the unstability results...





$$\left(\frac{1}{-\xi - \sqrt{\xi^2 - 1}}\right) = 1.865 \qquad \xi = -1.2$$

$$1 - 2S^2 = -1.205 < -1.2 = \xi$$

$$\omega = \frac{\pi}{\Delta t}$$

FDTD Yee schema + LDE

- FDTD most commonly applied to Maxwell's curl equations
- Usual to write the curl equation in a vacuum (with ε_0 and μ_0)
- Condense the *plasma physics* in the density of current **J**
- This results in a system coupling Maxwell equations with a LDE for **J=σE**

$$
\begin{aligned}
-\varepsilon_0 \partial_t \mathbf{E} + \nabla \times \mathbf{H} &= \mathbf{J} \\
\mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} &= 0 \\
\partial_t \mathbf{J} &= \varepsilon_0 \omega_p^2 \mathbf{E} - \nu \mathbf{J} + \omega_c \mathbf{b} \times \mathbf{J}
\end{aligned}
$$

- The curl equations are discretized following Yee schema
- A time integrator is used to solve **J** — the sensitive part...

Field components

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \frac{\sigma}{\varepsilon_0}E_x - \frac{1}{\varepsilon_0}J_x$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_0}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) - \frac{\sigma}{\varepsilon_0}E_y - \frac{1}{\varepsilon_0}J_y$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - \frac{\sigma}{\varepsilon_0}E_z - \frac{1}{\varepsilon_0}J_z$$

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu_0}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \frac{\sigma^\star}{\mu_0}H_x$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) - \frac{\sigma^\star}{\mu_0}H_y$$

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) - \frac{\sigma^\star}{\mu_0}H_z$$

Field components Bérenger PML formulation

$$\frac{\partial E_{xy}}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial (H_{zx}+H_{zy})}{\partial y} - \frac{\sigma_y}{\varepsilon_0}E_{xy} - \frac{1}{\varepsilon_0}J_{xy}$$

$$\frac{\partial E_{xz}}{\partial t} = -\frac{1}{\varepsilon_0}\frac{\partial (H_{yz}+H_{yx})}{\partial z} - \frac{\sigma_z}{\varepsilon_0}E_{xz} - \frac{1}{\varepsilon_0}J_{xz}$$

$$\frac{\partial E_{yz}}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial (H_{xy}+H_{xz})}{\partial z} - \frac{\sigma_z}{\varepsilon_0}E_{yz} - \frac{1}{\varepsilon_0}J_{yz}$$

$$\frac{\partial E_{yx}}{\partial t} = -\frac{1}{\varepsilon_0}\frac{\partial (H_{zx}+H_{zy})}{\partial x} - \frac{\sigma_x}{\varepsilon_0}E_{yx} - \frac{1}{\varepsilon_0}J_{yx}$$

$$\frac{\partial E_{zx}}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial (H_{yz}+H_{yx})}{\partial x} - \frac{\sigma_x}{\varepsilon_0}E_{zx} - \frac{1}{\varepsilon_0}J_{zx}$$

$$\frac{\partial E_{zy}}{\partial t} = -\frac{1}{\varepsilon_0}\frac{\partial (H_{xy}+H_{xz})}{\partial y} - \frac{\sigma_y}{\varepsilon_0}E_{zy} - \frac{1}{\varepsilon_0}J_{zy}$$

$$\frac{\partial H_{xy}}{\partial t} = -\frac{1}{\mu_0}\frac{\partial (E_{zx}+E_{zy})}{\partial y} - \frac{\sigma_y^\star}{\mu_0}H_{xy}$$

$$\frac{\partial H_{xz}}{\partial t} = \frac{1}{\mu_0}\frac{\partial (E_{yz}+E_{yx})}{\partial z} - \frac{\sigma_z^\star}{\mu_0}H_{xz}$$

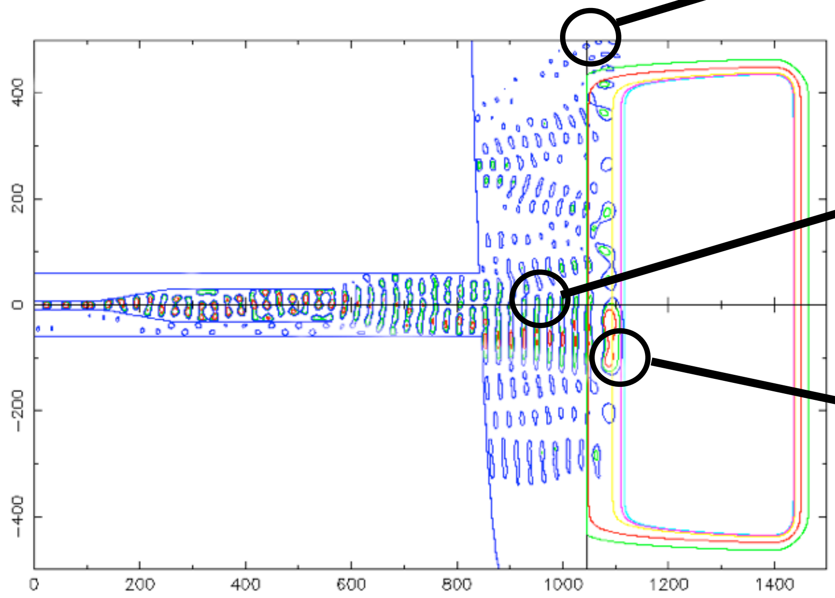
$$\frac{\partial H_{yz}}{\partial t} = -\frac{1}{\mu_0}\frac{\partial (E_{xy}+E_{xz})}{\partial z} - \frac{\sigma_z^\star}{\mu_0}H_{yz}$$

$$\frac{\partial H_{yx}}{\partial t} = \frac{1}{\mu_0}\frac{\partial (E_{zx}+E_{zy})}{\partial x} - \frac{\sigma_x^\star}{\mu_0}H_{yx}$$

$$\frac{\partial H_{zx}}{\partial t} = -\frac{1}{\mu_0}\frac{\partial (E_{yz}+E_{yx})}{\partial x} - \frac{\sigma_x^\star}{\mu_0}H_{zx}$$

$$\frac{\partial H_{zy}}{\partial t} = \frac{1}{\mu_0}\frac{\partial (E_{xy}+E_{xz})}{\partial y} - \frac{\sigma_y^\star}{\mu_0}H_{zy}$$

Maxwell Curl Equations+LDE



PML

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} - \sigma^\star \mathbf{H}$$

Vacuum

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

Plasma

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} + \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} - \sigma^\star \mathbf{H}$$

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \varepsilon_0 \omega_p^2 \mathbf{E} - \nu \mathbf{J} + \omega_c \times \mathbf{J}$$

NB.: With plasma movement, a correction is formally needed

Classic discretization

$$\varepsilon_0 \frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t} - \nabla \times \mathbf{H}^{n+1/2} = -\mathbf{J}^{n+1}$$

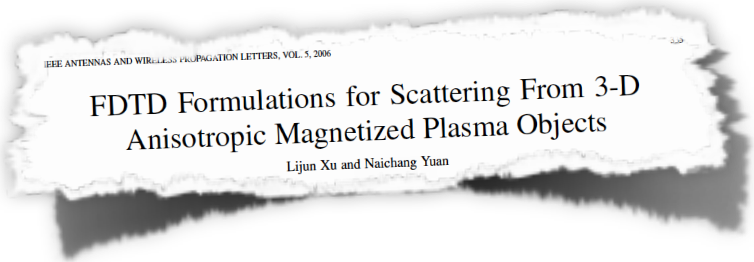
$$\mu_0 \frac{\mathbf{H}^{n+1/2} - \mathbf{H}^{n-1/2}}{\Delta t} + \nabla \times \mathbf{E}^n = 0$$

$$\frac{\mathbf{J}^{n+1/2} - \mathbf{J}^{n-1/2}}{\Delta t} = \varepsilon_0 \omega_p^2 \mathbf{E}^n + \omega_c \mathbf{b} \times \frac{\mathbf{J}^{n+1/2} + \mathbf{J}^{n-1/2}}{2}.$$

To be solved with an **efficient** and **stable** numerical schema...

A kernel, proposed by Lijun Xu and Naichang Yuan provides efficiency and *stability*

Lijun Xu and Naichang Yuan, IEEE Antennas and Wireless Propagation Letters, Vol. 5, 2006, pp 335-338


IEEE ANTENNAS AND WIRELESS PROPAGATION LETTERS, VOL. 5, 2006

FDTD Formulations for Scattering From 3-D Anisotropic Magnetized Plasma Objects

Lijun Xu and Naichang Yuan

For **REFMULX** and **REFMULF** the XY kernel was modified by us: **MXYK**

Searching for new solutions: Després-Pinto Kernel (DPK)

Implementing a new kernel proposed by B. Després and M. Campos Pinto

- Changes time centering of of J
- Relates **J** and **E** in a Crank-Nicolson way

$$
\begin{aligned}
\varepsilon_0 \frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t} - \mathrm{M}\mathbf{H}^{n+1/2} &= -\frac{\mathbf{J}^{n+1} + \mathbf{J}^n}{2} \\
\mu_0 \frac{\mathbf{H}^{n+1/2} - \mathbf{H}^{n-1/2}}{\Delta t} &= -\mathrm{M}^T \mathbf{E}^n \\
\frac{\mathbf{J}^{n+1} - \mathbf{J}^n}{\Delta t} &= \varepsilon_0 S^n(\omega_p^2) \frac{\mathbf{E}^n + \mathbf{E}^{n+1}}{2} + S^n(\omega_c) \mathbf{b} \wedge_h^n \frac{\mathbf{J}^{n+1} + \mathbf{J}^n}{2},
\end{aligned}
$$

- Operators M and M^T implement **∇×H** and **∇×E** on a Yee cell.
- S(u) an operator implements the multiplication of a vectorial field by a scalar u
- **b**$\wedge_h$ an operator approximating the cross product **b×** on staggered grids

Model and 2D geometry for o-mode

Plasma stationary on wave-time reference ($\tau_{plm} \gg T_{wav}$)

Ions considered motionless ($\omega_{ci} \ll \omega_{wav}$)

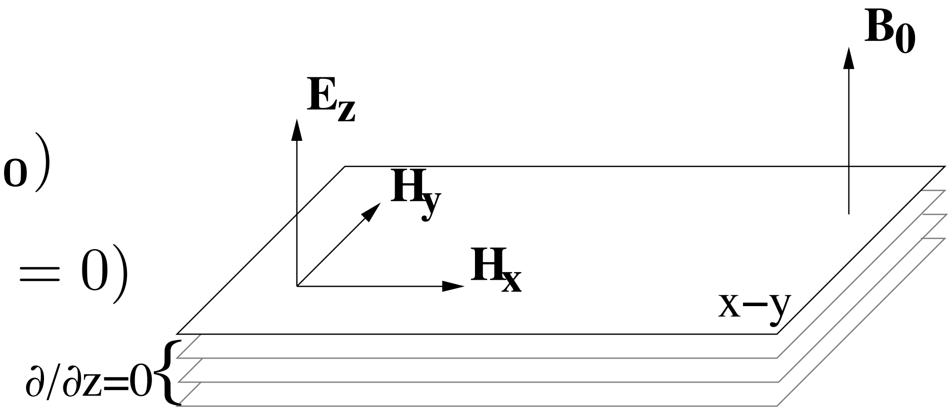
Thermal electron velocity smaller than phase velocity ($v_{th} \ll v_{ph}$)

Propagation in x–y plane

Static magnetic field along z ($\mathbf{B_0}$)

No gradients along z axis ($\partial/\partial z = 0$)

O-mode propagation ($\mathbf{E}//\mathbf{B_0}$)



O-mode/TMz mode

For the TMz mode (O-mode)

$$\mu_0 \frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y}$$

$$\mu_0 \frac{\partial H_y}{\partial t} = +\frac{\partial E_z}{\partial x}$$

$$\varepsilon_0 \frac{\partial E_z}{\partial t} = -\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x} - J_z$$

$$\frac{\mathrm{d}J_z}{\mathrm{d}t} + \nu J_z = \varepsilon_0 \omega_p^2 E_z$$

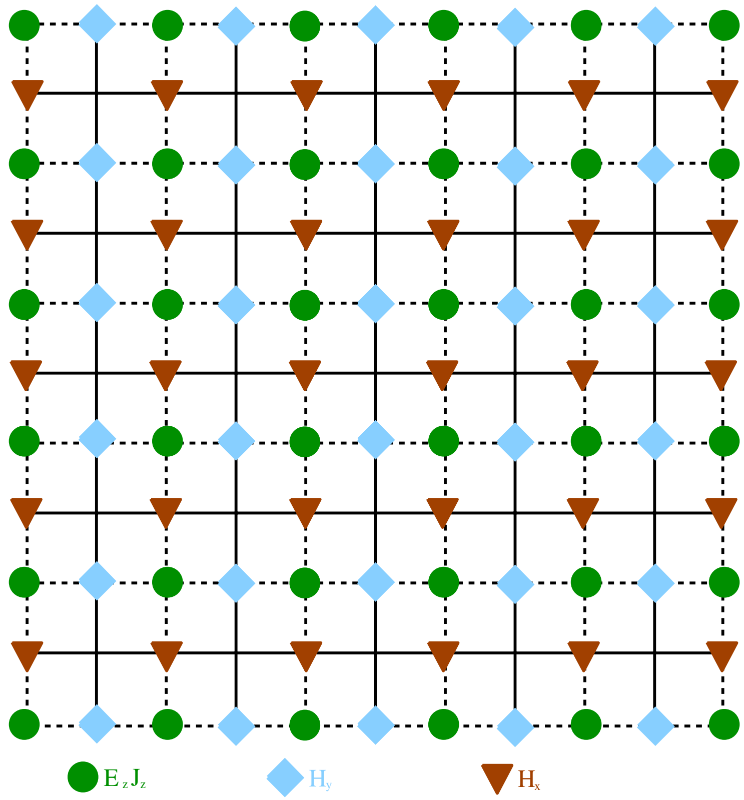
For the TMz mode (O-mode) in the PML

$$\mu_0 \frac{\partial H_x}{\partial t} + \sigma^\star H_x = -\frac{\partial E_z}{\partial y}$$

$$\mu_0 \frac{\partial H_y}{\partial t} + \sigma^\star H_y = +\frac{\partial E_z}{\partial x}$$

$$\varepsilon_0 \frac{\partial E_z}{\partial t} + \sigma E_z = -\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x}$$

TMz grid (O-mode)

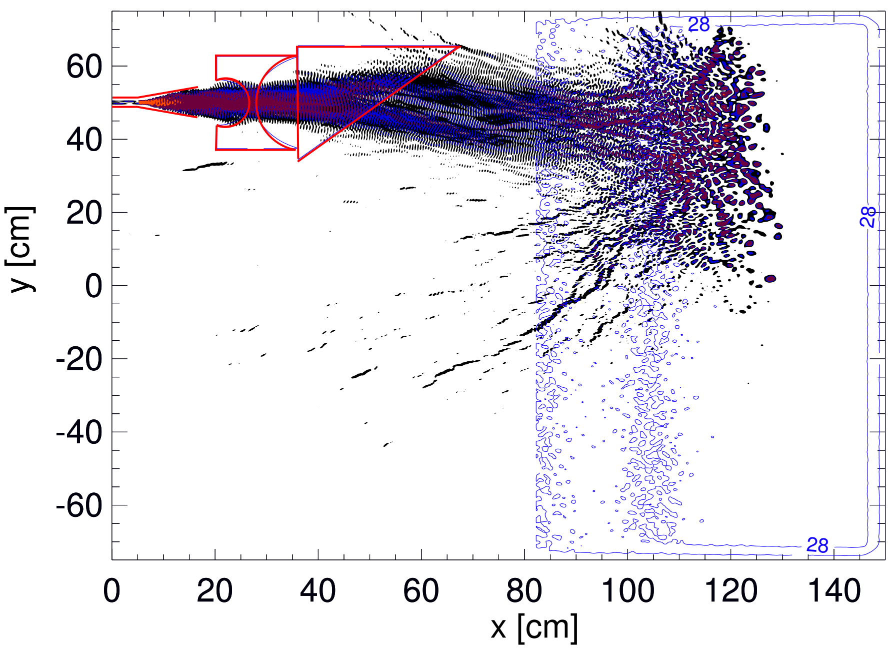


Example of a REFMUL simulation

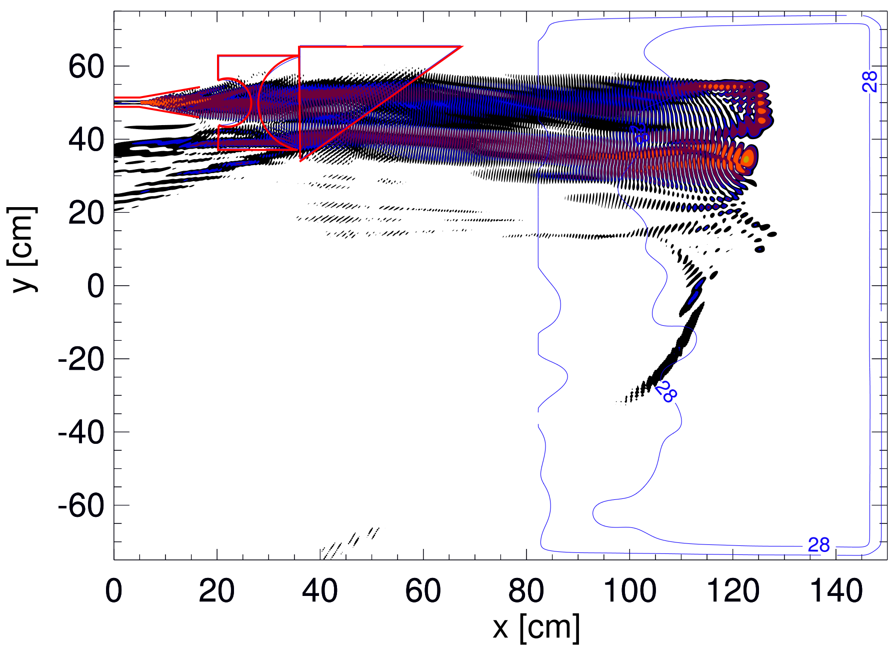
Forward Scattering

Forward-scattering under high level of turbulence or long propagation paths can induce significative effects and impose a signature on the Doppler reflectometry response.

Bragg backscattering (a) *mainly in the vicinity of the oblique cutoff for fluctuations having a resonant wavenumber given by* $\boldsymbol{k_{Bragg} = 2k_0 \sin(\alpha)}$.

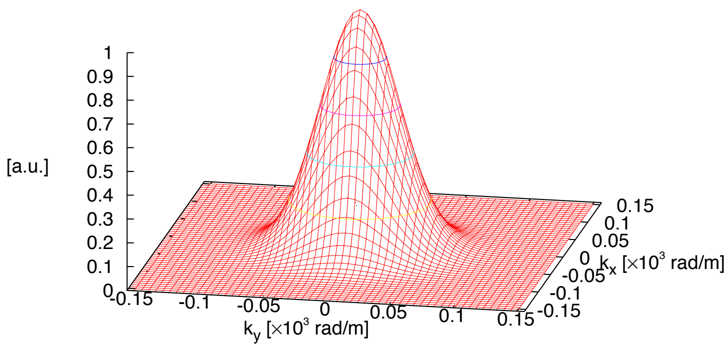


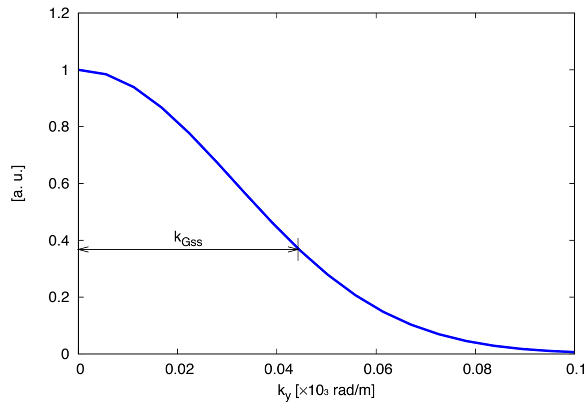
Cumulative forward-scattering (b), which becomes efficient for $\boldsymbol{k_{FS} < 2k_{Airy}}$



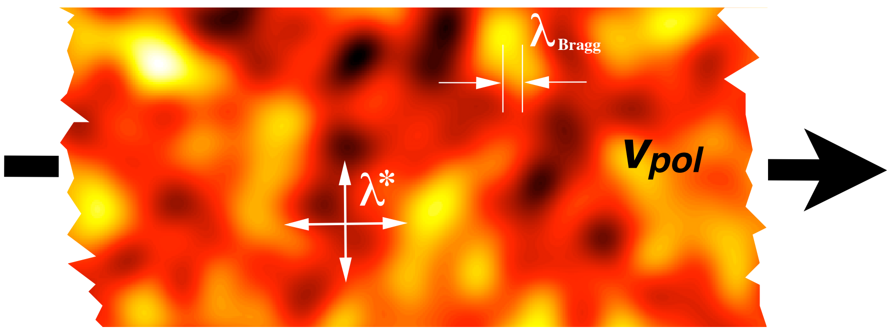
Model for turbulence

$$\delta n_{e_{TRB}} = \sum_{i=i_m}^{i_M} \sum_{j=j_m}^{j_M} A(i,j) \times \cos\left[k_x(i)x + k_y(j)y + \varphi(i,j)\right]$$





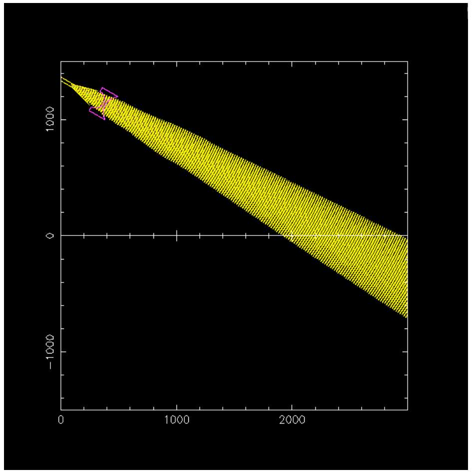
The amplitude spectrum modeling plasma turbulence was chosen to allow only forward scattering. There are no *structures* with a Bragg ressonant size λ_{Bragg}.



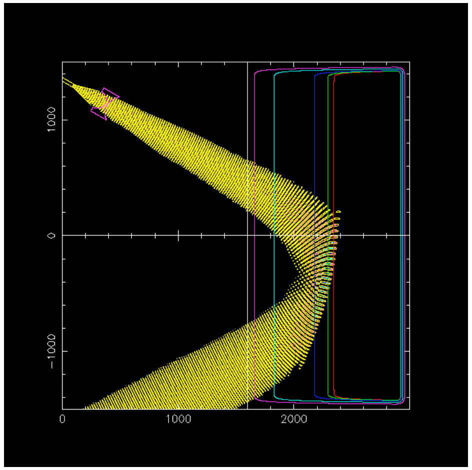
Plasma slab model 1,353 × 3,000 grid points
$f = 40$ GHz ($n_c \approx 2 \cdot 10^{19} m^{-3}$)
RMS (@35 GHz. $\delta n_e/n_e(x_{35GHz}) \times 100\%$)
$v_y = 22.5 \cdot 10^6\ m \cdot s^{-1}$, i.e. $v_y/c = 7.5\%$.
L_m=30,000 grid points
N = 800,000 iterations

Snapshots of runs (α=30º)

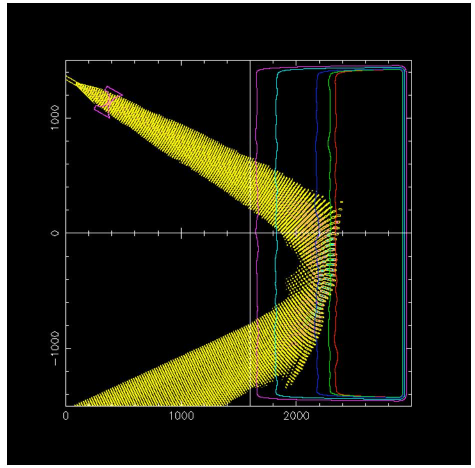
vacuum



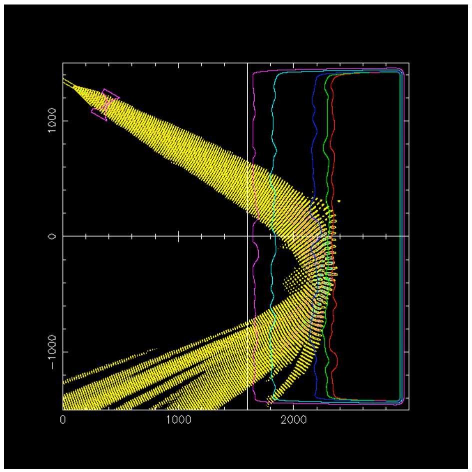
δn_e=0



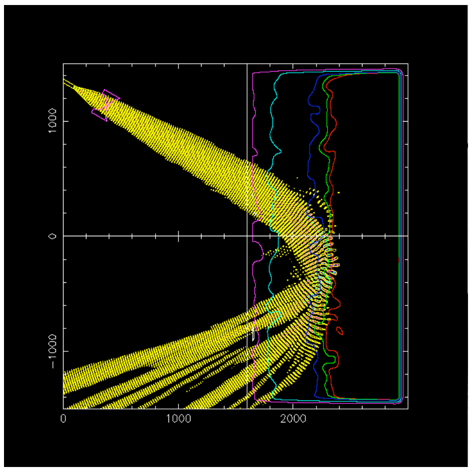
δn_e=1.0%



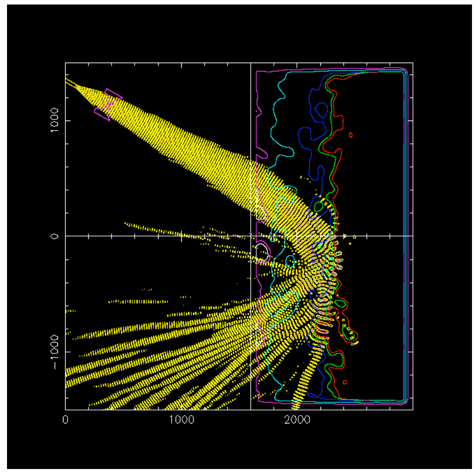
δn_e=2.5%



δn_e=5.0%

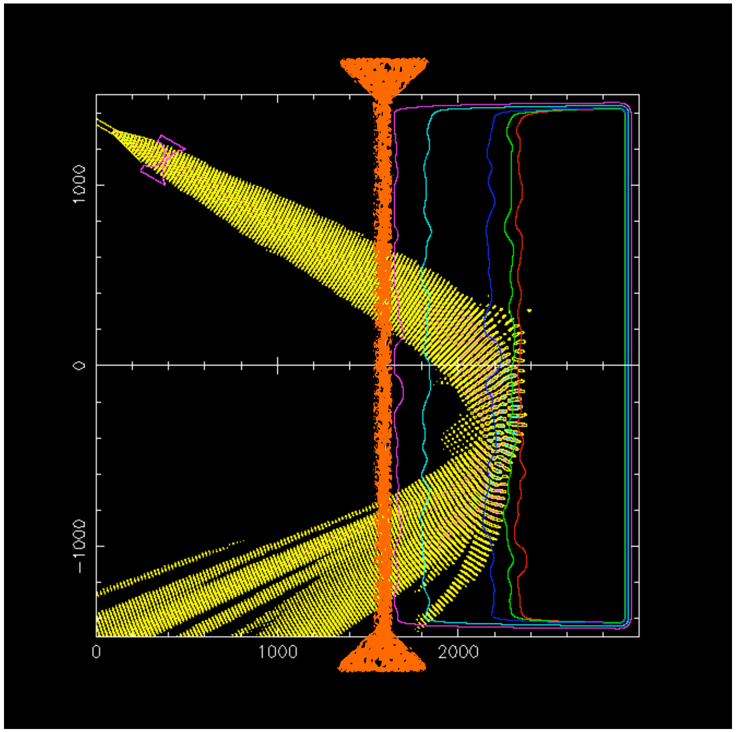


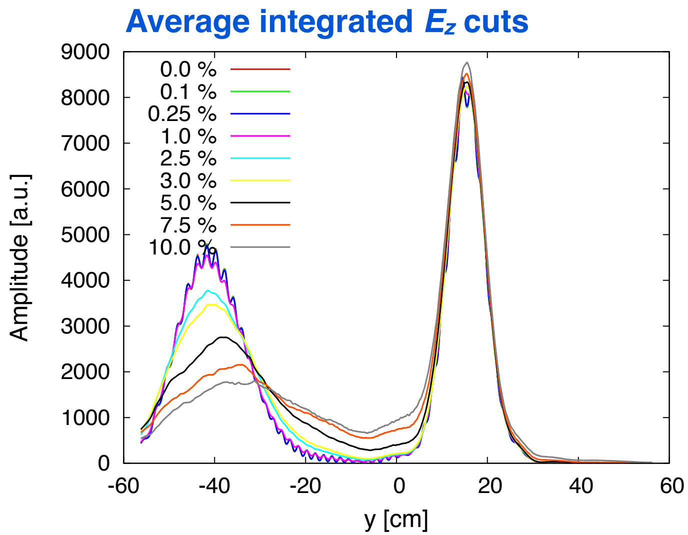
δn_e=10.0%

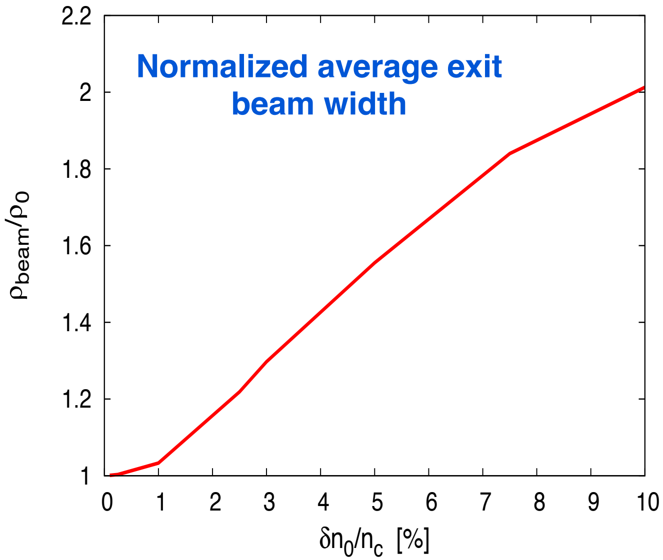


Beam enlargement

$$\int |E_{z_{edge}}|^2 \mathrm{d}t$$

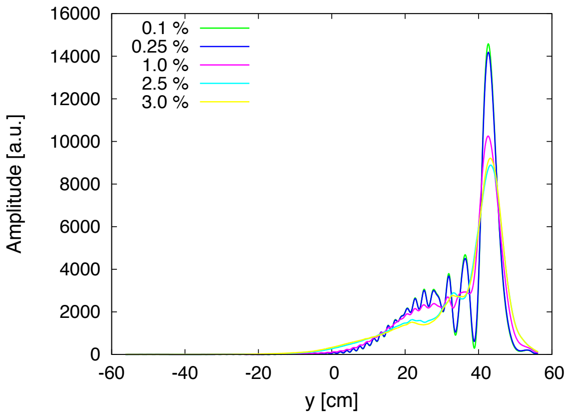




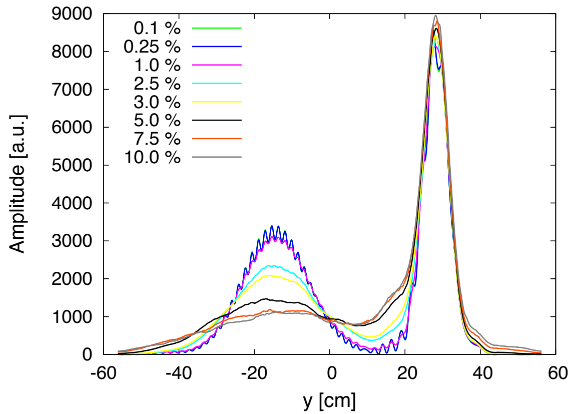


Runs at different angles

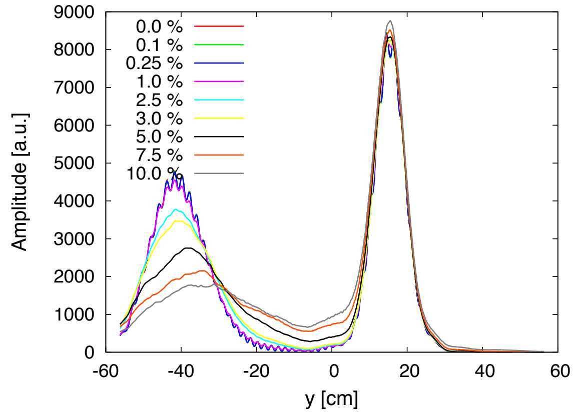
Average integrated E_z cuts α=7°



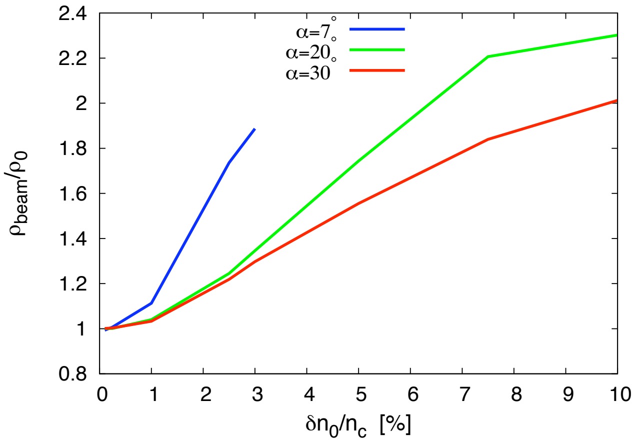
Average integrated E_z cuts α=20°



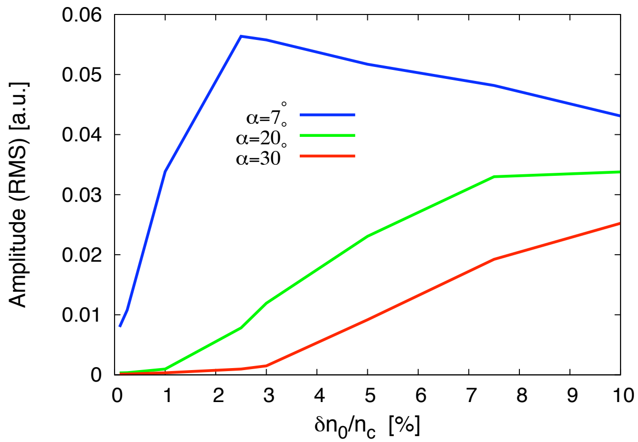
Average integrated E_z cuts α=30°



Normalized average exit beam width



RMS at the antenna



Model and 2D geometry for x-mode

Plasma stationary on wave-time reference ($\tau_{plm} \gg T_{wav}$)

Ions considered motionless ($\omega_{ci} \ll \omega_{wav}$)

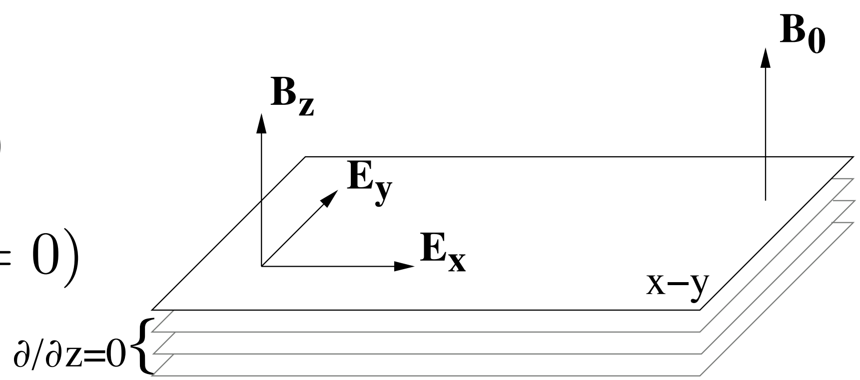
Thermal electron velocity smaller than phase velocity ($v_{th} \ll v_{ph}$)

Propagation in x–y plane

Static magnetic field along z ($\mathbf{B_0}$)

No gradients along z axis ($\partial/\partial z = 0$)

X-mode propagation ($\mathbf{H}//\mathbf{B_0}$)



TMz and TEz modes

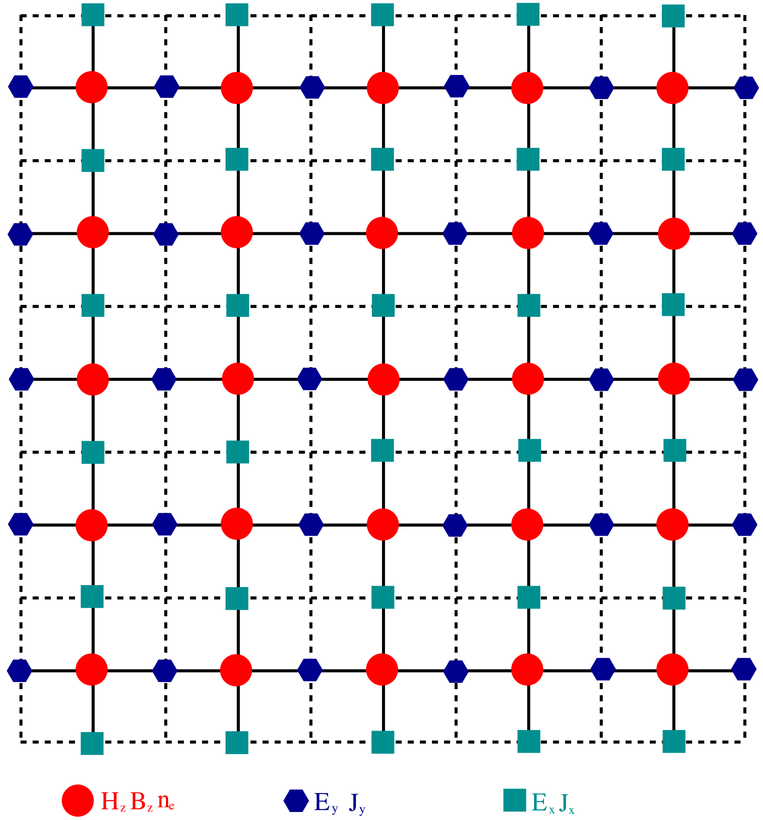
For the TEz mode (X-mode)

$$\varepsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - J_x$$
$$\varepsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} - J_y$$
$$\mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$
$$\frac{\mathrm{d}J_x}{\mathrm{d}t} + \nu J_x = \varepsilon_0 \omega_p^2 E_x - \omega_z \times J_y$$
$$\frac{\mathrm{d}J_y}{\mathrm{d}t} + \nu J_y = \varepsilon_0 \omega_p^2 E_y + \omega_z \times J_x$$

For the TEz mode (X-mode) in the PML

$$\varepsilon_0 \frac{\partial E_x}{\partial t} + \sigma E_x = \frac{\partial H_z}{\partial y}$$
$$\varepsilon_0 \frac{\partial E_y}{\partial t} + \sigma E_y = -\frac{\partial H_z}{\partial x}$$
$$\mu_0 \frac{\partial H_z}{\partial t} + \sigma^\star H_z = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$

TEz grid (X-mode)

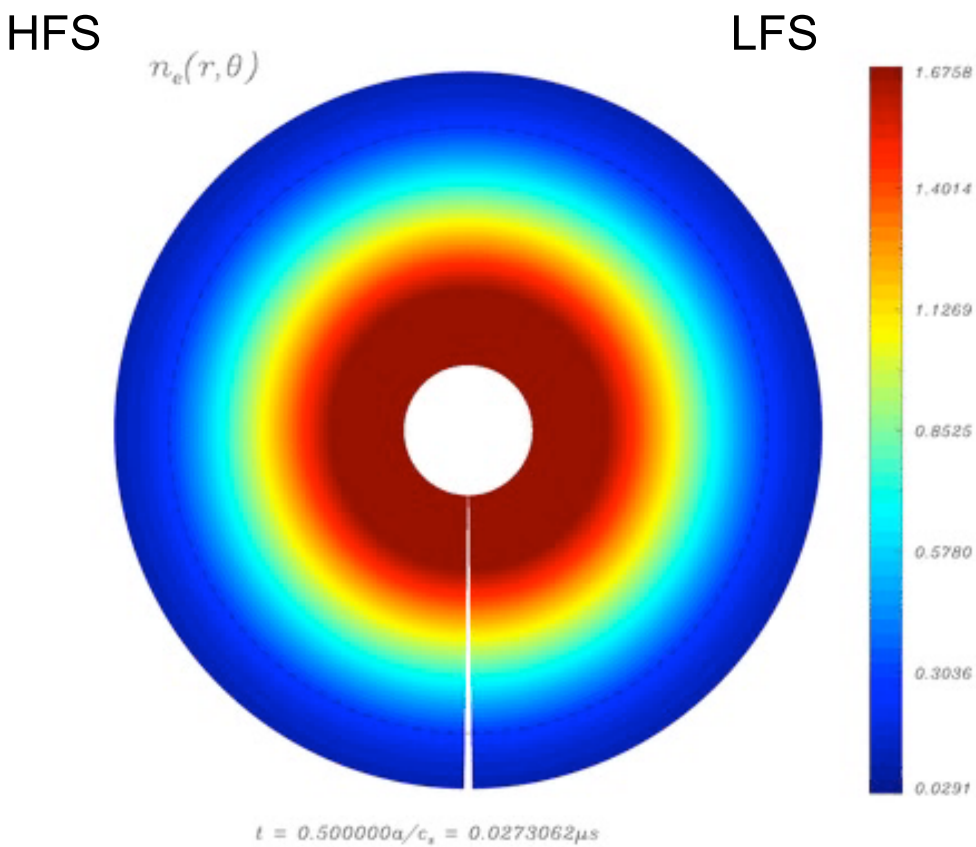


Example of a REFMULX simulation

- 6-moment electromagnetic gyrofluid model for each plasma species:
  - density and ∥ velocity
  - ∥ and ⊥ temperature and heat flux
- Polarisation and induction: field equations
- Dependent variables include the zonal profile:
  - e.g. $n_e \rightarrow \tilde{n}_e + n_0(x)$
- Includes a scrape-off layer (SOL) next to the edge region
- Dependent variables include the zonal profile:
  - radial (x) dependence retained (no flux tube approximation)
  - reduced-MHD equilibrium part: self consistent Shafranov shift

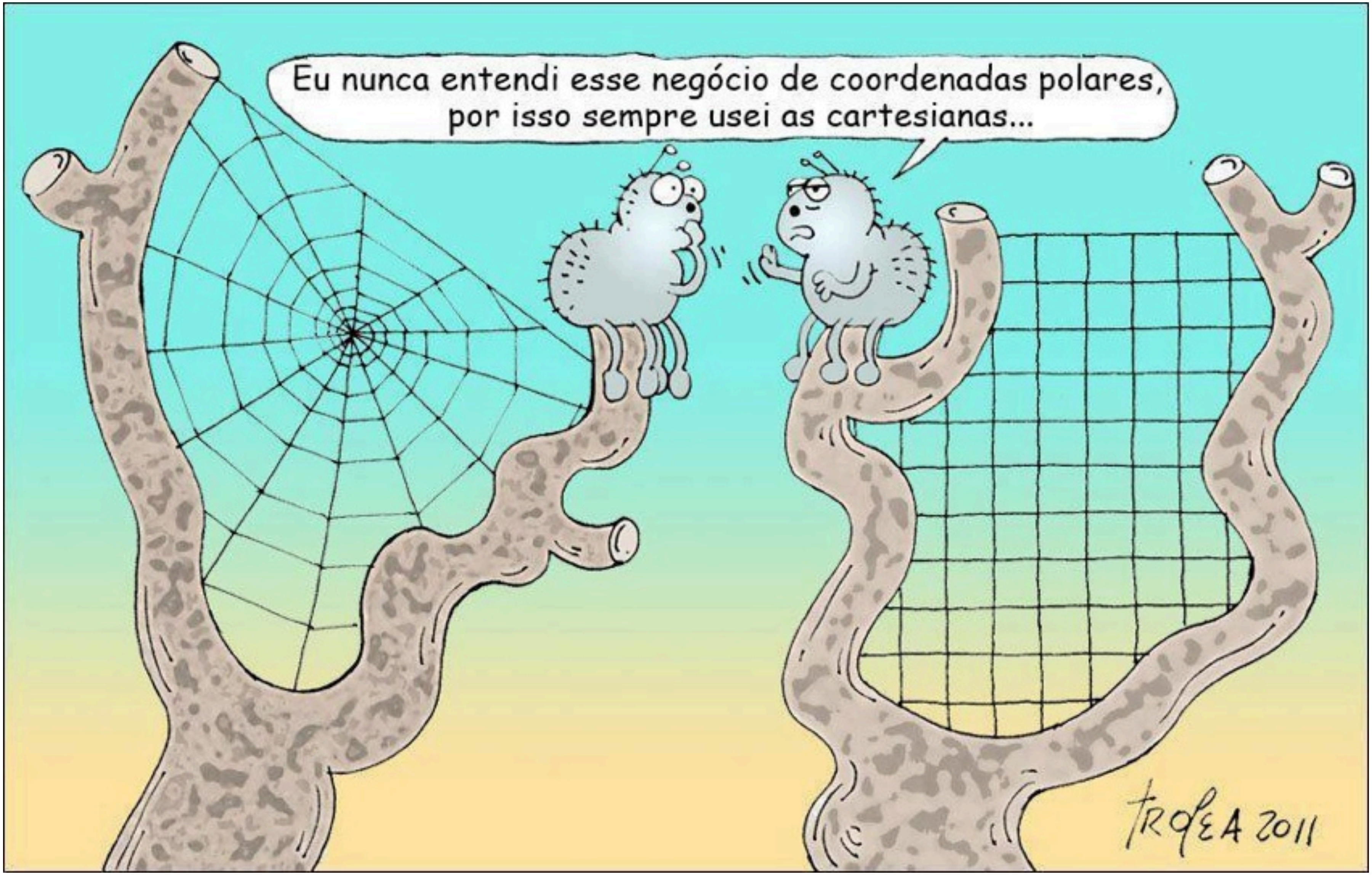
Gyrofluid models are suitable to study tokamak turbulence while demanding moderate computational resources

An illustrative case



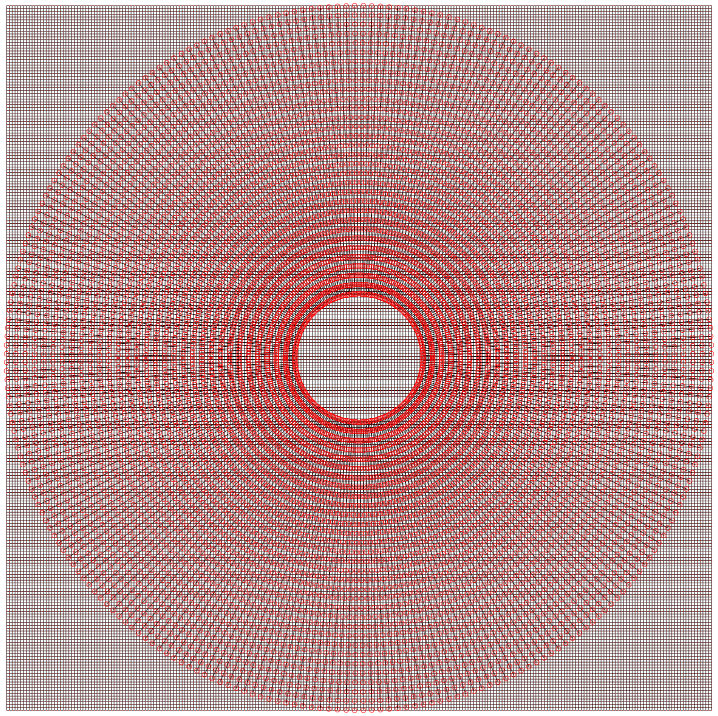
Much smaller sized than ASDEX Upgrade

Different code domains



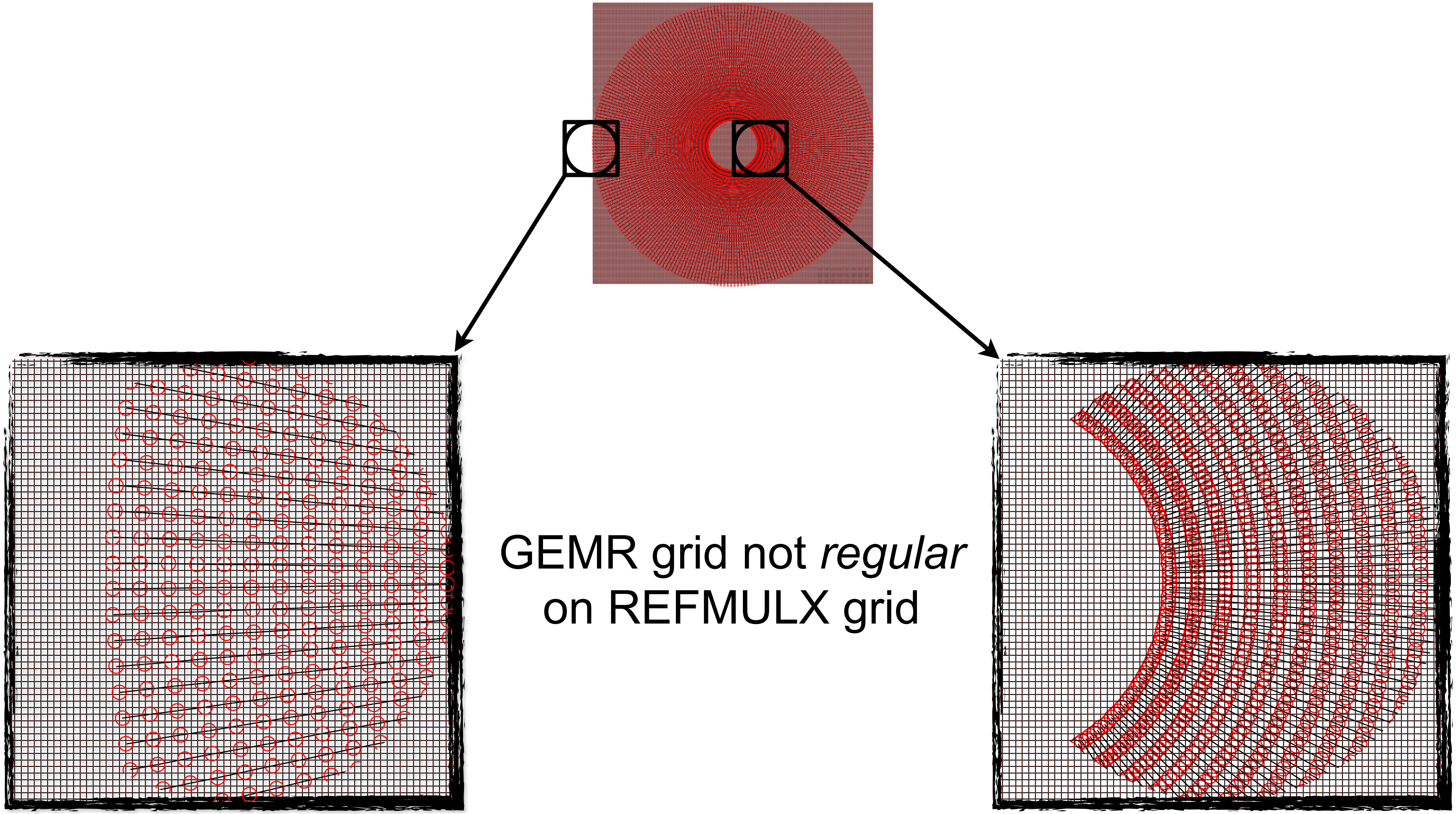
Different code domains

GEMR defined in polar coordinates (r,θ)

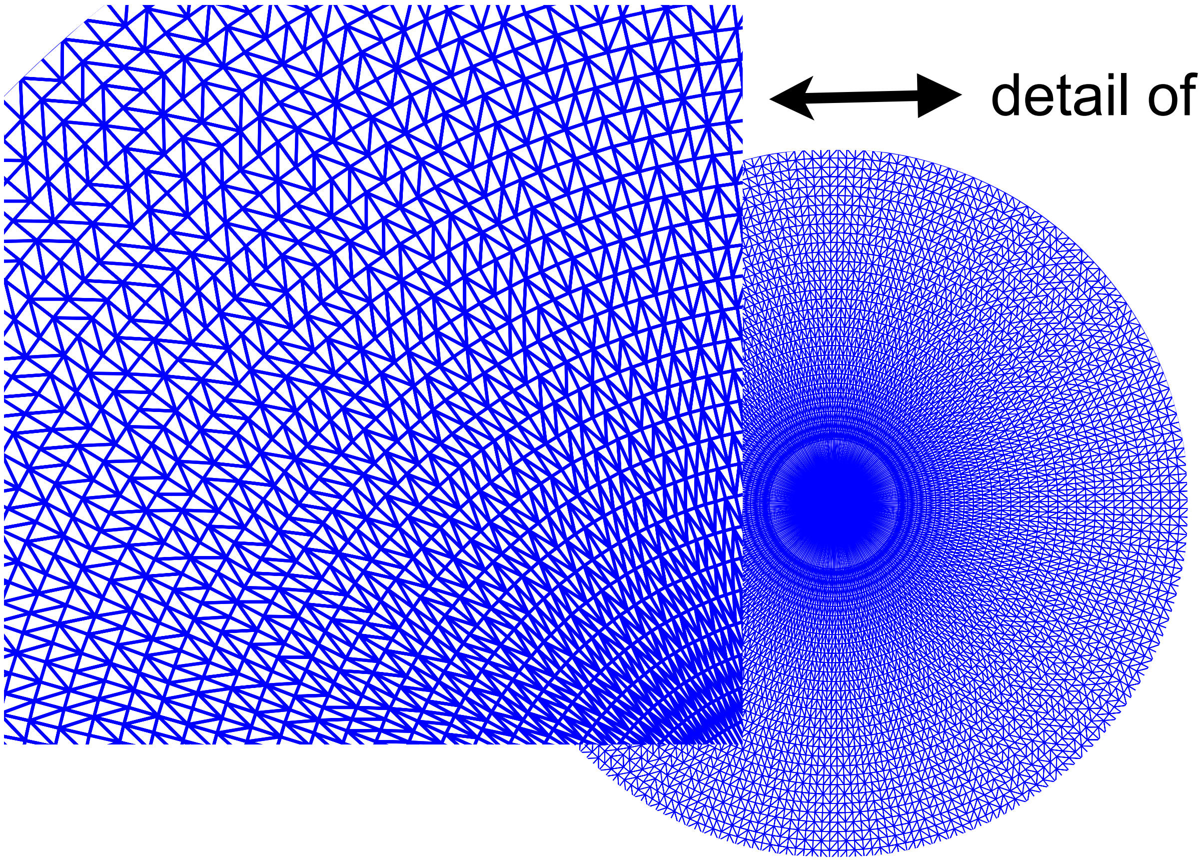


REFMULX defined in Cartesian coordinates (x,y)

Different code domains



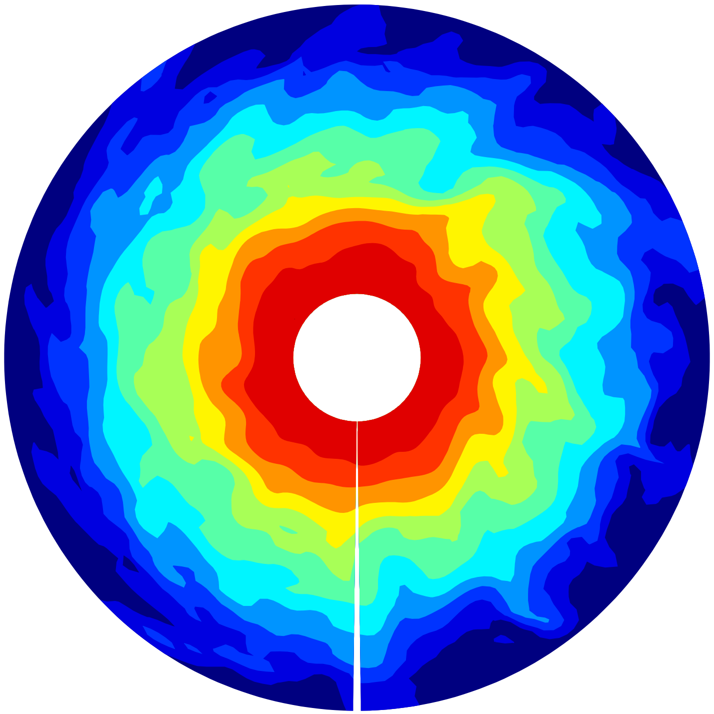
GEMR grid not *regular* on REFMULX grid



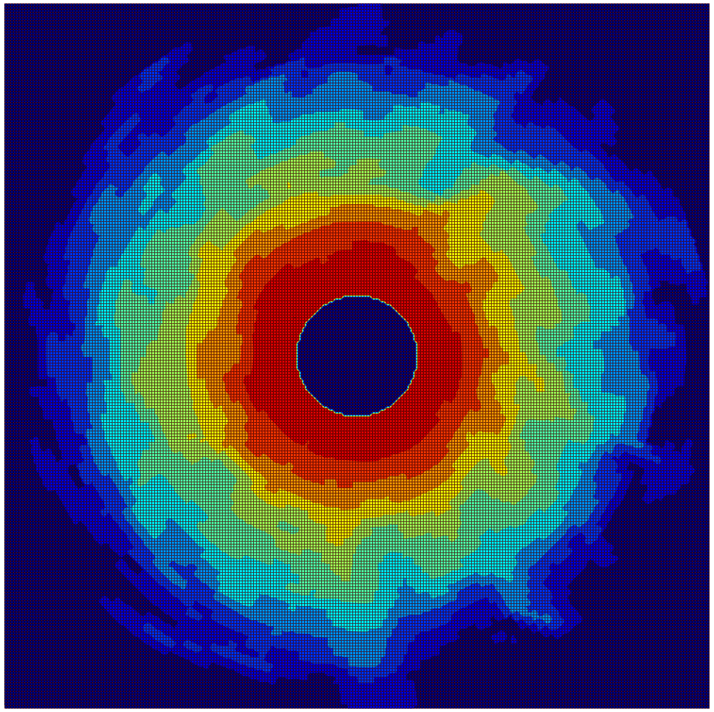
detail of the triangulation

A Delaunay interpolation technique is used

From polar to Cartesian...

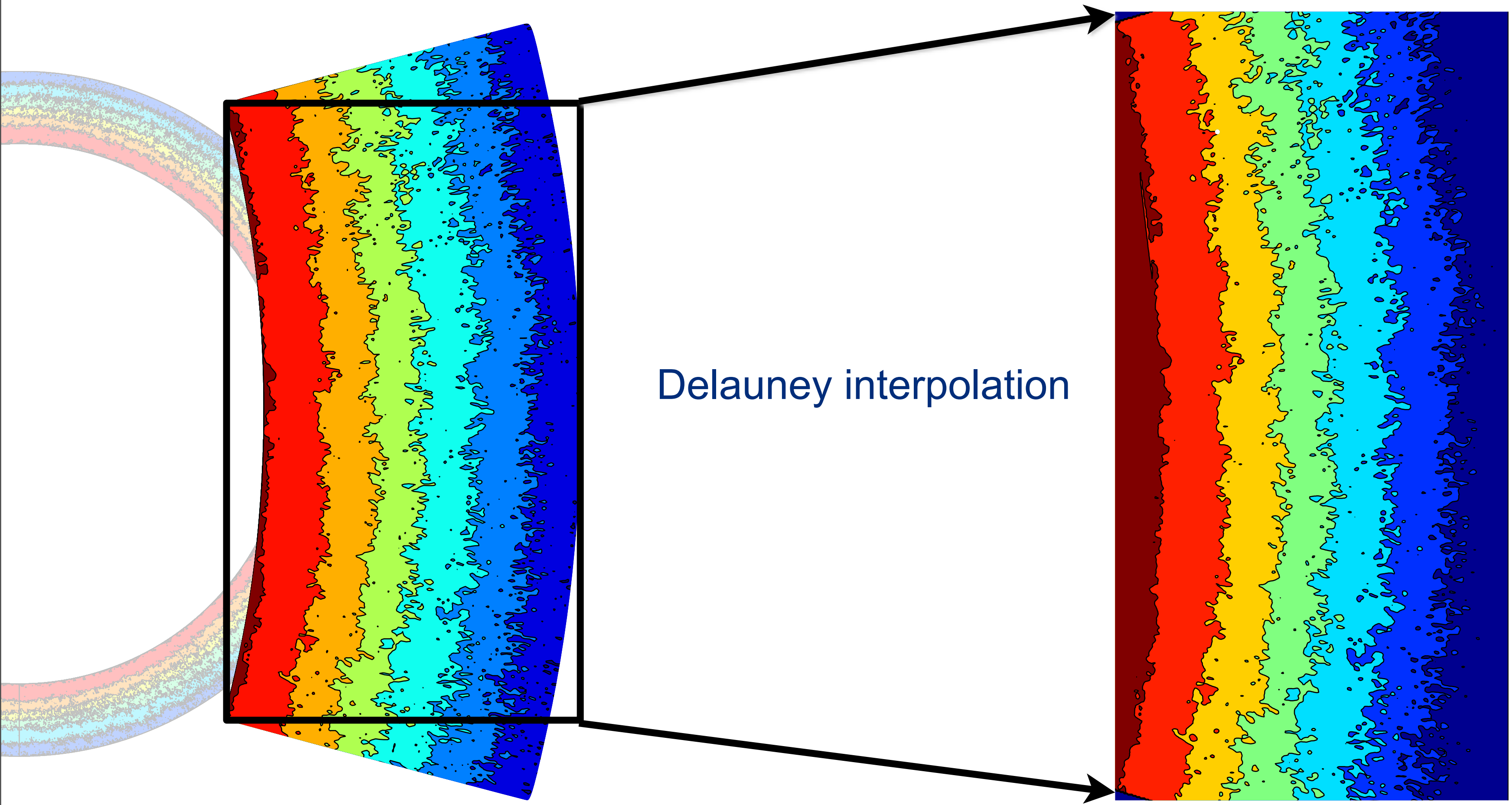


GEMR output



REFMULX input

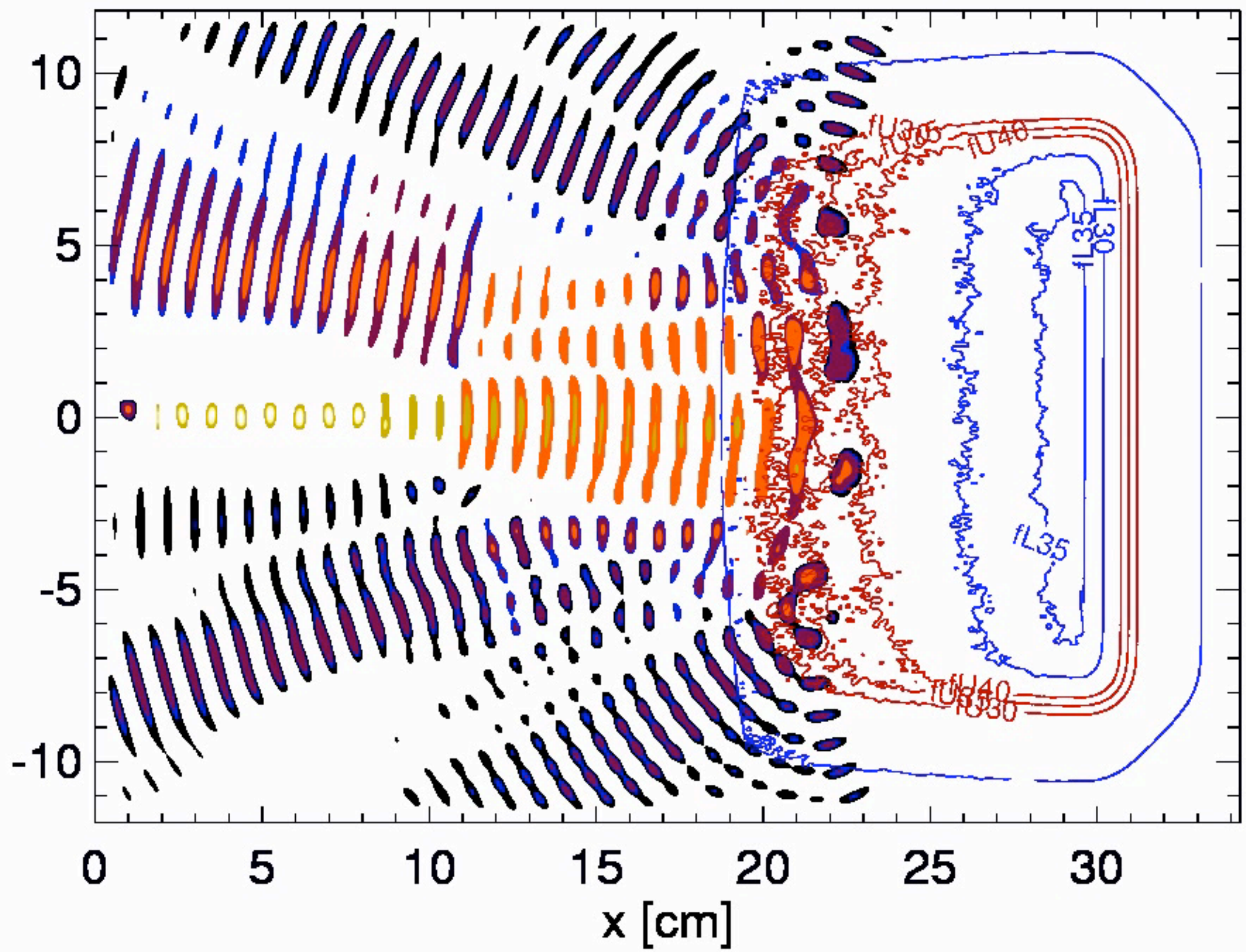
Towards an ASDEX Upgrade configuration...



Local set of physical quantities: typical ASDEX Upgrade edge (L-mode)

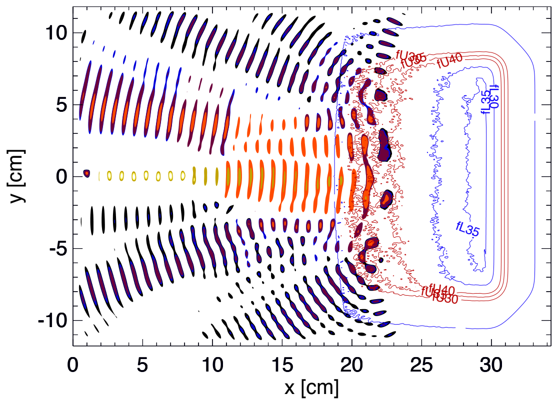
A run sample...

80,000 — 80,800 iterations shown

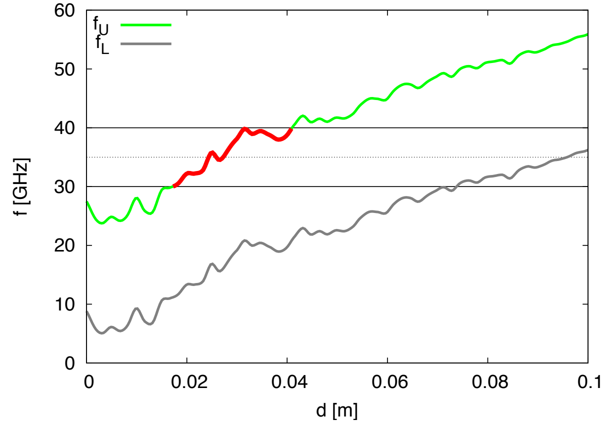


Results of the swept run:

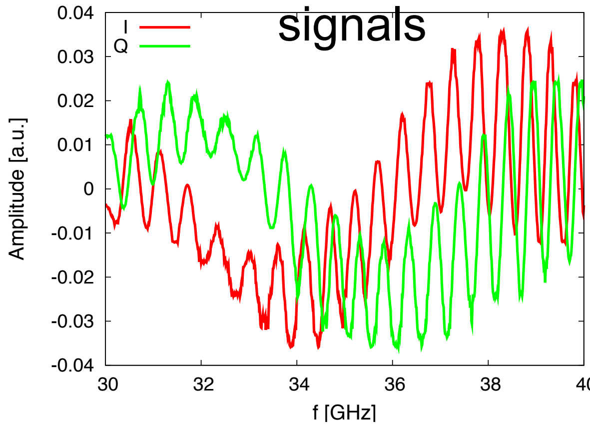
Field map



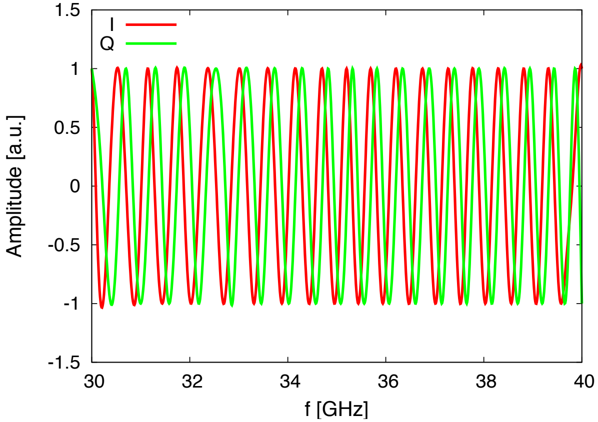
Cutoffs



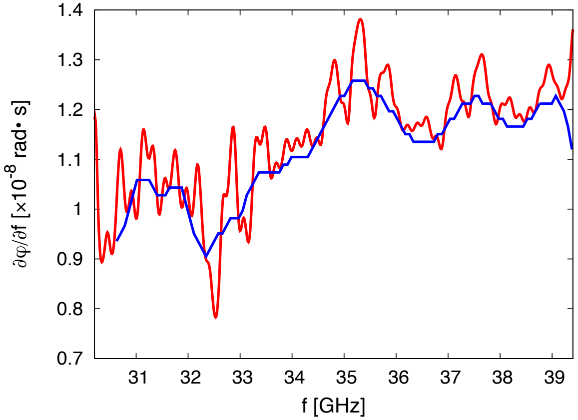
I/Q reflectometry signals



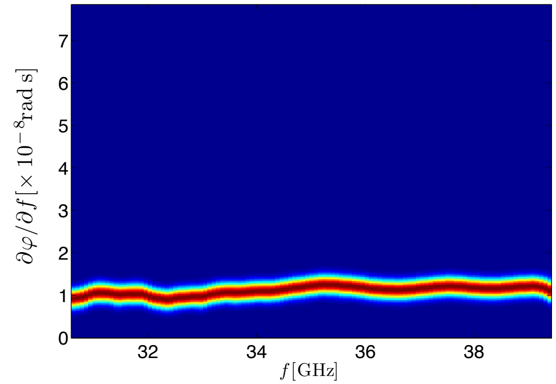
I/Q normalized signals



Phase derivative



SFFT

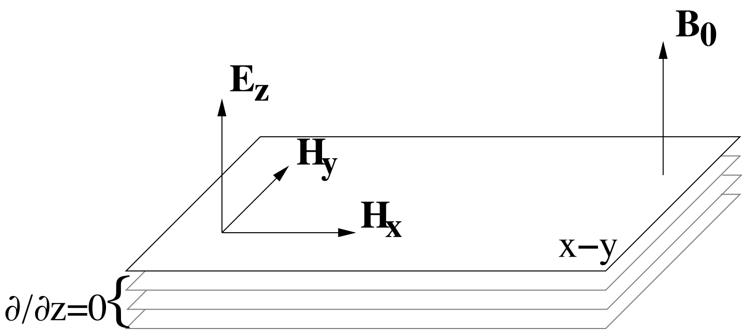


Recalling 2D geometry for o-mode and x-mode

Propagation in 2D (x-y plane)

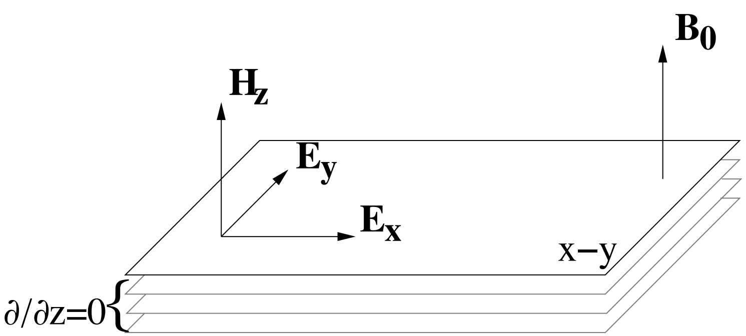
External magnetic field, $\mathbf{B_0}$ along z

No gradients along z axis (∂/∂z=0)



Transverse magnetic mode TMz (E_z, H_x, H_y)

O-mode propagation ($\mathbf{E}$//$\mathbf{B_0}$)



Transverse electric mode TEz (H_z, E_x, E_y)

X-mode propagation ($\mathbf{H}$//$\mathbf{B_0}$)

Recalling equations for TMz and TEz modes

For the TMz mode (O-mode)

$$\mu_0 \frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\partial y}$$

$$\mu_0 \frac{\partial H_y}{\partial t} = +\frac{\partial E_z}{\partial x}$$

$$\varepsilon_0 \frac{\partial E_z}{\partial t} = -\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x} - J_z$$

$$\frac{\mathrm{d}J_z}{\mathrm{d}t} + \nu J_z = \varepsilon_0 \omega_p^2 E_z$$

For the TMz mode (O-mode) in the PML

$$\mu_0 \frac{\partial H_x}{\partial t} + \sigma^\star H_x = -\frac{\partial E_z}{\partial y}$$

$$\mu_0 \frac{\partial H_y}{\partial t} + \sigma^\star H_y = +\frac{\partial E_z}{\partial x}$$

$$\varepsilon_0 \frac{\partial E_z}{\partial t} + \sigma E_z = -\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x}$$

For the TEz mode (X-mode)

$$\varepsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - J_x$$

$$\varepsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} - J_y$$

$$\mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$

$$\frac{\mathrm{d}J_x}{\mathrm{d}t} + \nu J_x = \varepsilon_0 \omega_p^2 E_x - \omega_z \times J_y$$

$$\frac{\mathrm{d}J_y}{\mathrm{d}t} + \nu J_y = \varepsilon_0 \omega_p^2 E_y + \omega_z \times J_x$$

For the TEz mode (X-mode) in the PML

$$\varepsilon_0 \frac{\partial E_x}{\partial t} + \sigma E_x = \frac{\partial H_z}{\partial y}$$

$$\varepsilon_0 \frac{\partial E_y}{\partial t} + \sigma E_y = -\frac{\partial H_z}{\partial x}$$

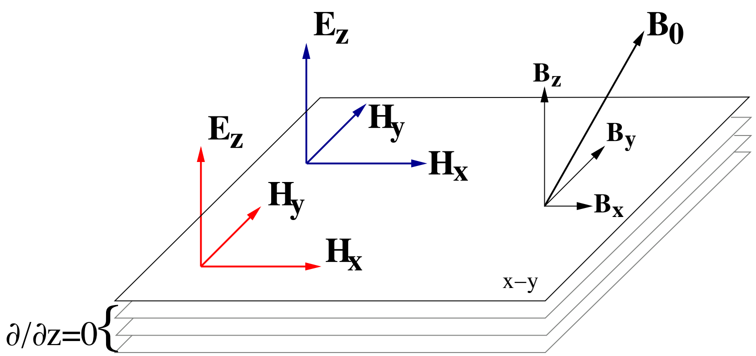
$$\mu_0 \frac{\partial H_z}{\partial t} + \sigma^\star H_z = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$

REFMULF 2D geometry

Propagation in 2D (x-y plane)

General external magnetic field, $\mathbf{B_0}=\mathbf{B_x}+\mathbf{B_y}+\mathbf{B_z}$

No gradients along z axis ($\partial/\partial z=0$)



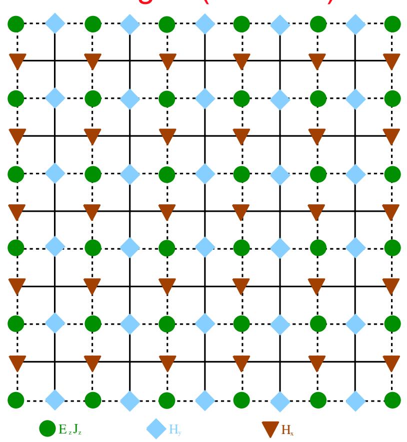
Transverse magnetic mode TMz (E_z, H_x, H_y)

O-mode propagation (**E**//**B**$_0$)

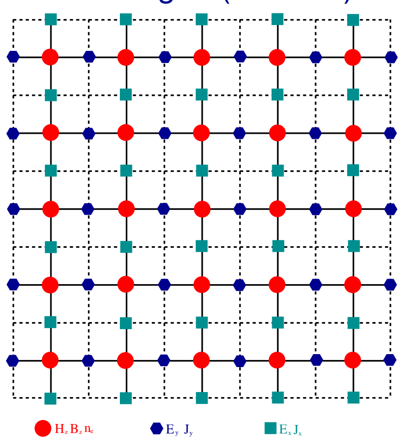
Transverse electric mode TEz (H_z, E_x, E_y)

X-mode propagation (**H**//**B**$_0$)

TMz grid (O-mode)



TEz grid (X-mode)



Maxwell curl equations can only couple TMz and TEz modes through $\partial/\partial z$

BUT

Coupling though J is possible

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \varepsilon_0\omega_p^2\mathbf{E} - \nu\mathbf{J} + \omega_c \times \mathbf{J}$$

$$\frac{\mathrm{d}J_x}{\mathrm{d}t} + \nu J_x = \varepsilon_0\omega_p^2 E_x + \omega_y J_z - \omega_z J_y$$

$$\frac{\mathrm{d}J_y}{\mathrm{d}t} + \nu J_y = \varepsilon_0\omega_p^2 E_y + \omega_z J_x - \omega_x J_z$$

B_x or B_y couples TMz plane to TEz

$$\frac{\mathrm{d}J_z}{\mathrm{d}t} + \nu J_z = \varepsilon_0\omega_p^2 E_z + \omega_x J_y - \omega_y J_x$$

B_x or B_y couples TEz plane to TMz

Xu & Yu expression for J_x

$$
\begin{aligned}
C_{0x} &= 1 + \frac{\omega_x^2 \Delta t^2}{4} \\
C_{1x} &= 1 + \frac{\omega_y^2 \Delta t^2}{4} - \left(\frac{\omega_x^2 \omega_y^2 \Delta t^4}{16 C_{0x}} - \frac{\Delta t^2 \omega_z^2}{4 C_{0x}} \right) \\
J_x^{n+1/2}(i, j+1/2) &= \frac{1}{C_{1x}} \left[1 - \frac{\omega_y^2 \Delta t^2}{4} + \left(\frac{\omega_x^2 \omega_y^2 \Delta t^4}{16 C_{0x}} - \frac{\Delta t^2 \omega_z^2}{4 C_{0x}} \right) \right] J_x^{n-1/2}(i, j+1/2) + \\
&+ \frac{\varepsilon_0 \omega_p^2 \Delta t}{C_{1x}} \left[E_x^n(i, j+1/2) + \left(\frac{\omega_x \omega_y \Delta t^2}{4 C_{0x}} - \frac{\Delta t \omega_z}{2 C_{0x}} \right) E_y^n(i+1/2, j) \right] + \\
&+ \frac{\varepsilon_0 \omega_p^2 \Delta t^2}{2 C_{1x}} \left[\omega_y - \left(\frac{\omega_x^2 \omega_y \Delta t^2}{4 C_{0x}} - \frac{\Delta t \omega_z \omega_x}{2 C_{0x}} \right) \right] E_z^n(i-1/2, j-1/2) + \\
&+ \left(\frac{\omega_x \omega_y \Delta t^2}{4 C_{1x}} - \frac{\Delta t \omega_z}{2 C_{1x}} \right) \left[1 + \left(\frac{1}{C_{0x}} - \frac{\omega_x^2 \Delta t^2}{4 C_{0x}} \right) \right] J_y^{n-1/2}(i+1/2, j) + \\
&+ \frac{1}{C_{1x}} \left[\omega_y \Delta t - \frac{\omega_x \Delta t}{C_{0x}} \left(\frac{\omega_x \omega_y \Delta t^2}{4} - \frac{\Delta t \omega_z}{2} \right) \right] J_z^{n-1/2}(i-1/2, j-1/2)
\end{aligned}
$$

Xu & Yu expression for J_y

$$
\begin{aligned}
C_{0y} &= 1+\frac{\omega_y^2\Delta t^2}{4}\\
C_{1y} &= 1+\frac{\omega_z^2\Delta t^2}{4}-\left(\frac{\omega_y^2\omega_z^2\Delta t^4}{16C_{0y}}-\frac{\Delta t^2\omega_x^2}{4C_{0y}}\right)\\
J_y^{n+1/2}(i+1/2,j) &= \frac{1}{C_{1y}}\left[1-\frac{\omega_z^2\Delta t^2}{4}+\left(\frac{\omega_y^2\omega_z^2\Delta t^4}{16C_{0y}}-\frac{\Delta t^2\omega_x^2}{4C_{0y}}\right)\right]J_y^{n-1/2}(i+1/2,j)+\\
&+ \frac{\varepsilon_0\omega_p^2\Delta t}{C_{1y}}\left[E_y^n(i+1/2,j)+\left(\frac{\omega_y\omega_z\Delta t^2}{4C_{0y}}-\frac{\Delta t\omega_x}{2C_{0y}}\right)E_z^n(i-1/2,j-1/2)\right]+\\
&+ \frac{\varepsilon_0\omega_p^2\Delta t^2}{2C_{1y}}\left[\omega_z-\left(\frac{\omega_y^2\omega_z\Delta t^2}{4C_{0y}}-\frac{\Delta t\omega_x\omega_y}{2C_{0y}}\right)\right]E_x^n(i,j+1/2)+\\
&+ \left(\frac{\omega_y\omega_z\Delta t^2}{4C_{1y}}-\frac{\Delta t\omega_x}{2C_{1y}}\right)\left[1+\left(\frac{1}{C_{0y}}-\frac{\omega_y^2\Delta t^2}{4C_{0y}}\right)\right]J_z^{n-1/2}(i-1/2,j-1/2)+\\
&+ \frac{1}{C_{1y}}\left[\omega_z\Delta t-\frac{\omega_y\Delta t}{C_{0y}}\left(\frac{\omega_y\omega_z\Delta t^2}{4}-\frac{\Delta t\omega_x}{2}\right)\right]J_x^{n-1/2}(i,j+1/2)
\end{aligned}
$$

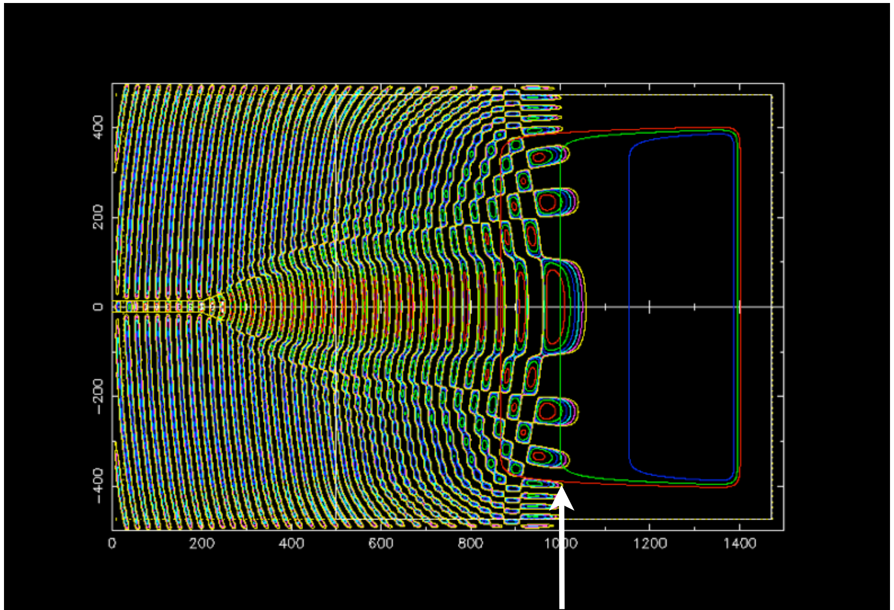
Xu & Yu expression for J_z

$$
\begin{aligned}
C_{0z} &= 1 + \frac{\omega_z^2 \Delta t^2}{4} \\
C_{1z} &= 1 + \frac{\omega_x^2 \Delta t^2}{4} - \left(\frac{\omega_z^2 \omega_x^2 \Delta t^4}{16 C_{0z}} - \frac{\Delta t^2 \omega_y^2}{4 C_{0z}} \right) \\
J_z^{n+1/2}(i-1/2, j-1/2) &= \frac{1}{C_{1z}} \left[1 - \frac{\omega_x^2 \Delta t^2}{4} + \left(\frac{\omega_z^2 \omega_x^2 \Delta t^4}{16 C_{0z}} - \frac{\Delta t^2 \omega_y^2}{4 C_{0z}} \right) \right] J_z^{n-1/2}(i-1/2, j-1/2) + \\
&+ \frac{\varepsilon_0 \omega_p^2 \Delta t}{C_{1z}} \left[E_z^n(i-1/2, j-1/2) + \left(\frac{\omega_z \omega_x \Delta t^2}{4 C_{0z}} - \frac{\Delta t \omega_y}{2 C_{0z}} \right) E_x^n(i, j+1/2) \right] + \\
&+ \frac{\varepsilon_0 \omega_p^2 \Delta t^2}{2 C_{1z}} \left[\omega_x - \left(\frac{\omega_z^2 \omega_x \Delta t^2}{4 C_{0z}} - \frac{\Delta t \omega_y \omega_z}{2 C_{0z}} \right) \right] E_y^n(i+1/2, j) + \\
&+ \left(\frac{\omega_z \omega_x \Delta t^2}{4 C_{1z}} - \frac{\Delta t \omega_y}{2 C_{1z}} \right) \left[1 + \left(\frac{1}{C_{0z}} - \frac{\omega_z^2 \Delta t^2}{4 C_{0z}} \right) \right] J_x^{n-1/2}(i, j+1/2) + \\
&+ \frac{1}{C_{1z}} \left[\omega_x \Delta t - \frac{\omega_z \Delta t}{C_{0z}} \left(\frac{\omega_z \omega_x \Delta t^2}{4} - \frac{\Delta t \omega_y}{2} \right) \right] J_y^{n-1/2}(i+1/2, j)
\end{aligned}
$$

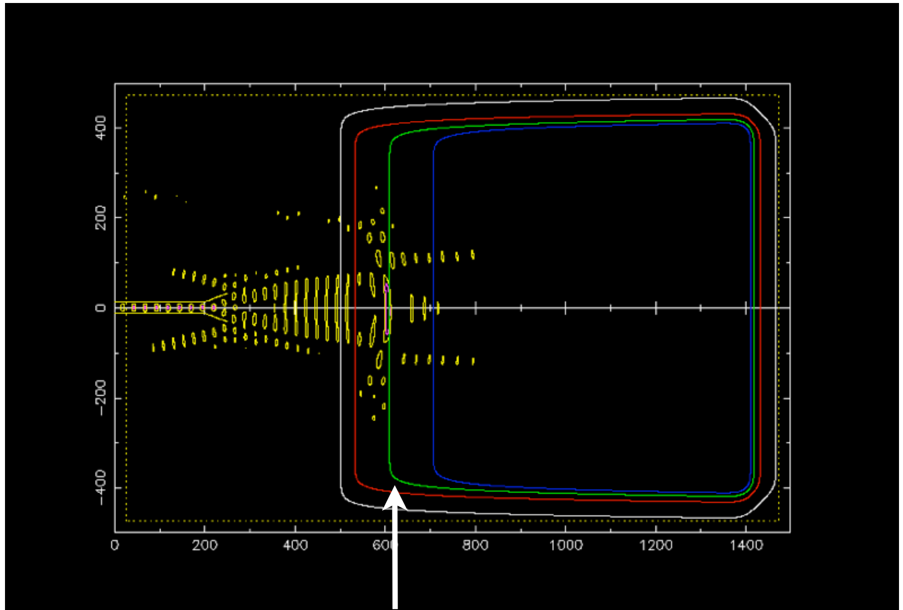
O-mode/TMz excitation

O-mode probing with f=35 GHz, B_z=0.95T, B_y=0.01T

O-mode probing, E_z field



X-mode mode coupling, H_z field



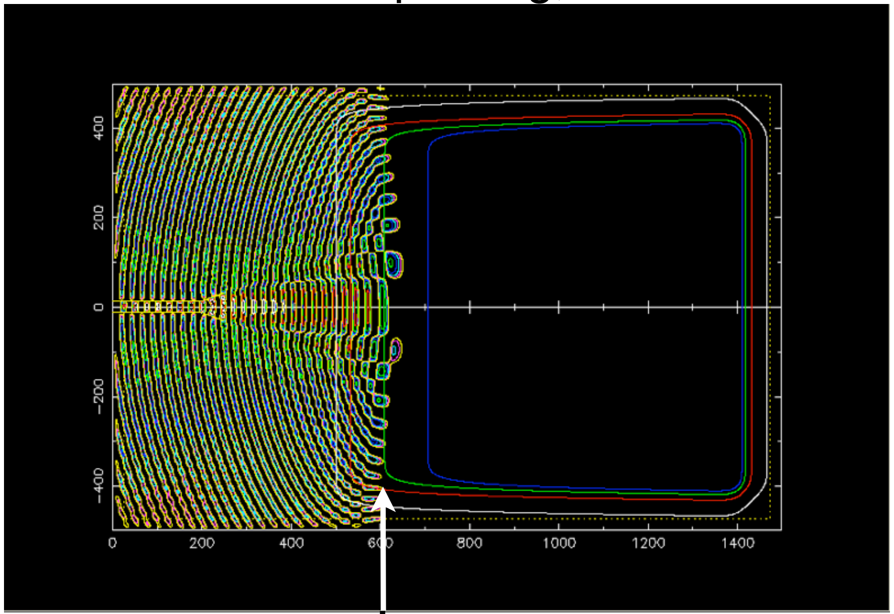
Plasma frequency cutoff f_p

Upper cutoff f_U

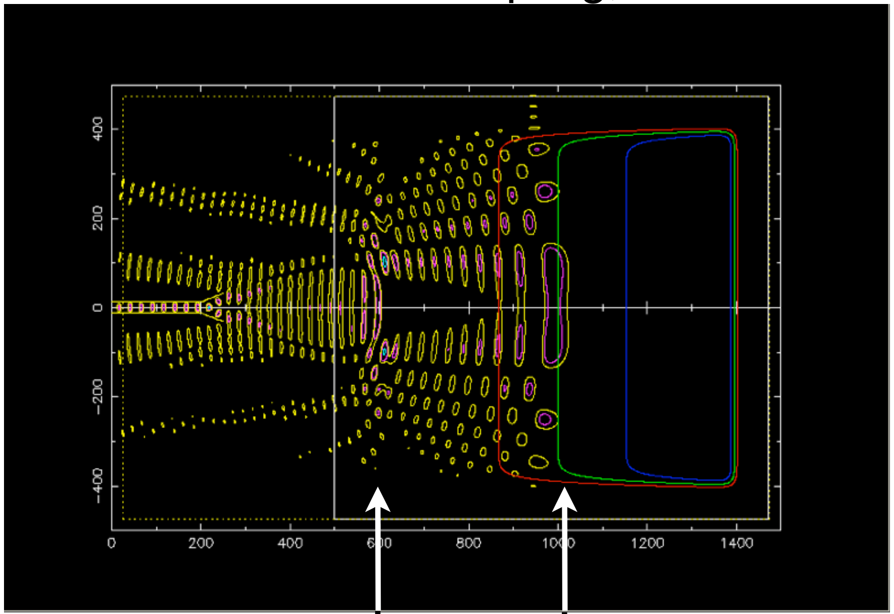
X-mode/TEz excitation

X-mode probing with f=35 GHz, Bz=0.95T, By=0.01T

X-mode probing, H_z field



O-mode mode coupling, E_z field



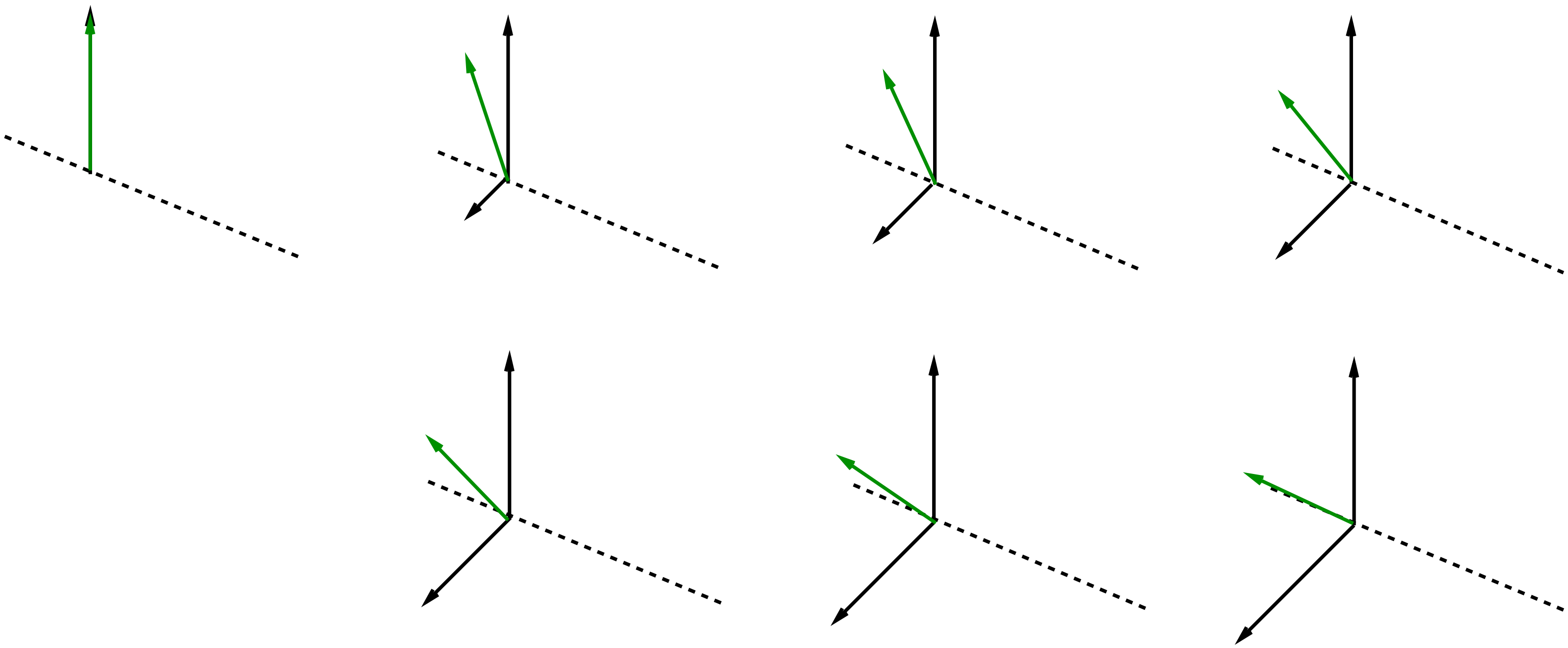
Upper cutoff f_U

Plasma frequency cutoff f_p

B_z constant, Increasing B_y

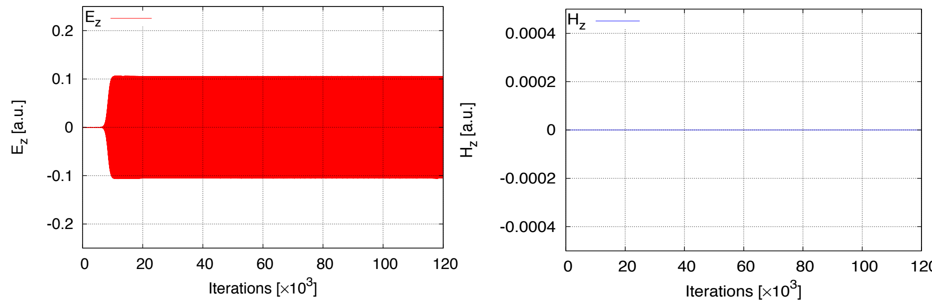
B_z = 0.9T

B_y = 0.0, 0.15, 0.30, 0.45, 0.60, 0.75, 0.9 T

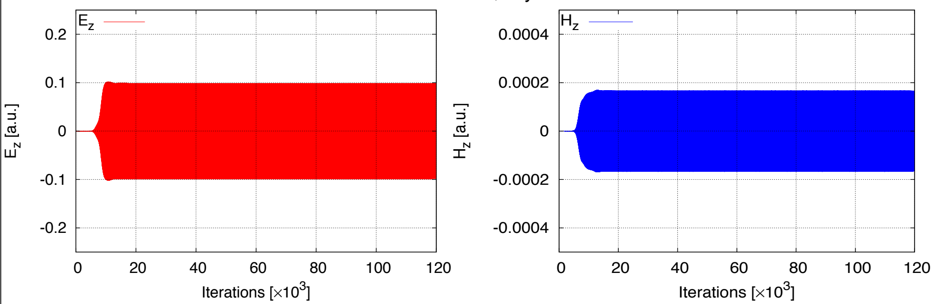


O-mode excitation E_z and x-mode coupled H_z signals

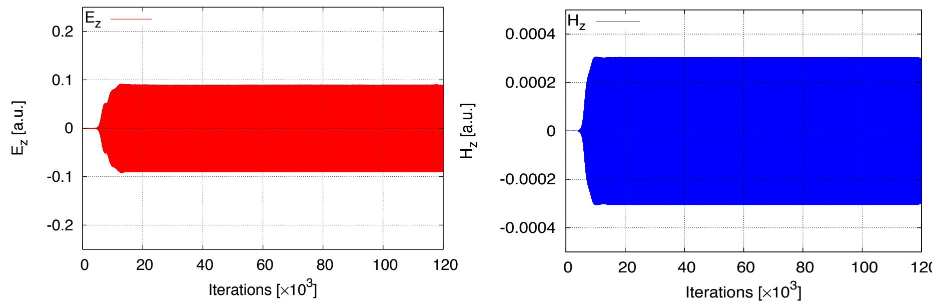
B_z=0.9T, B_y=0T



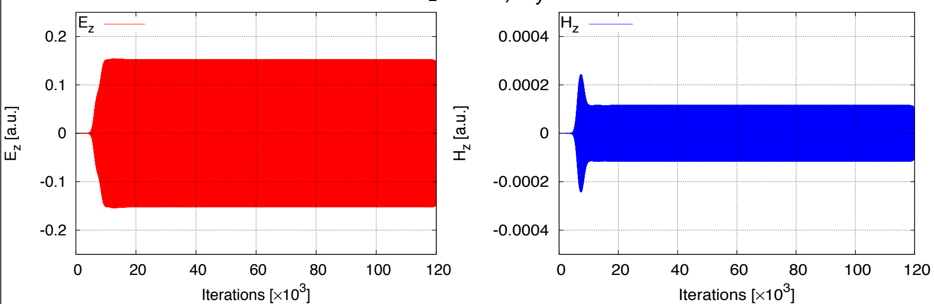
B_z=0.9T, B_y=0.3T



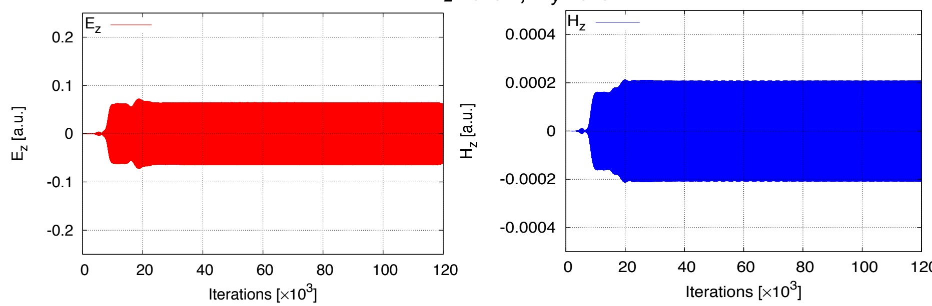
B_z=0.9T, B_y=0.6T



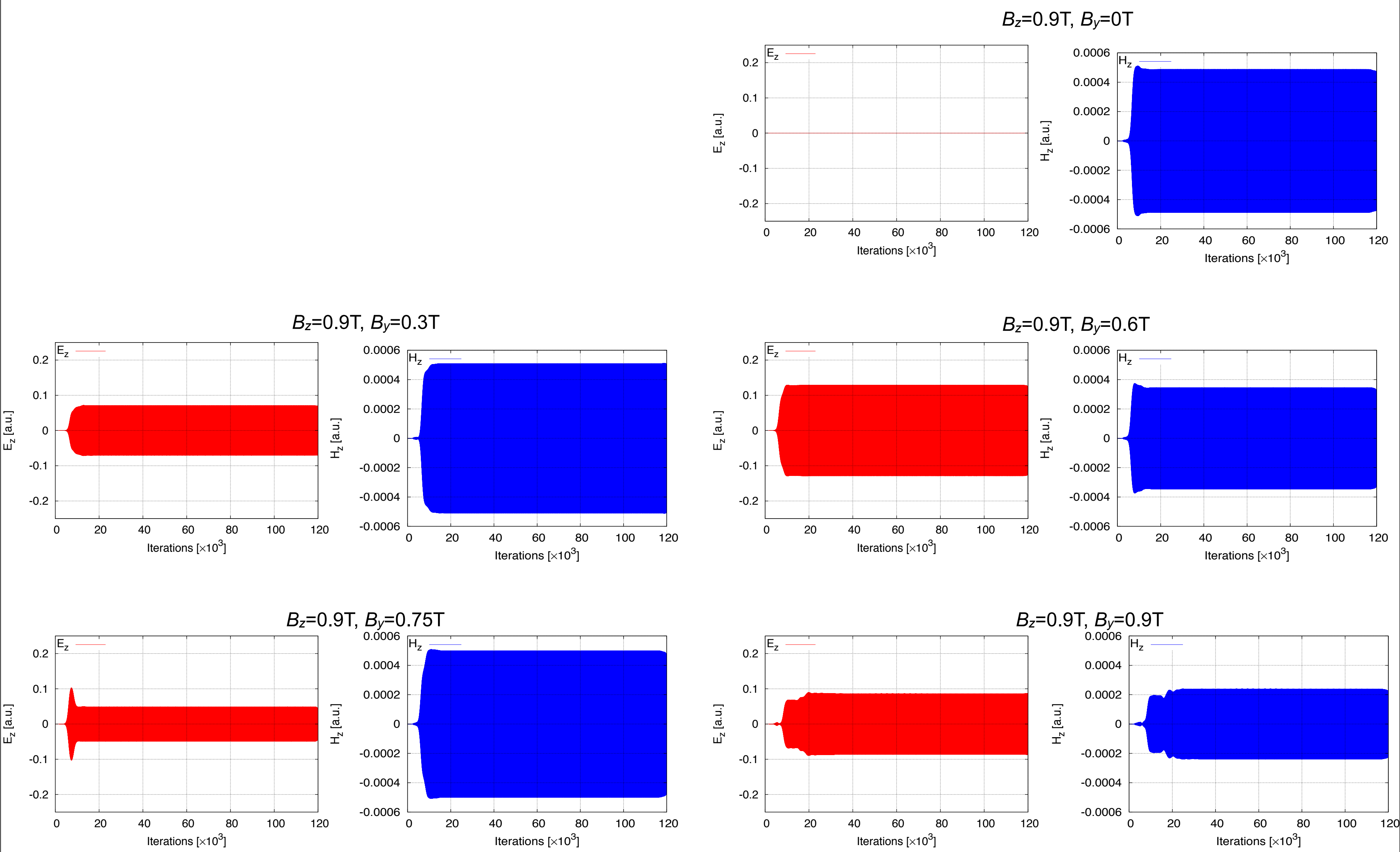
B_z=0.9T, B_y=0.75T



B_z=0.9T, B_y=0.9T



X-mode excitation H_z and o-mode coupled E_z signals

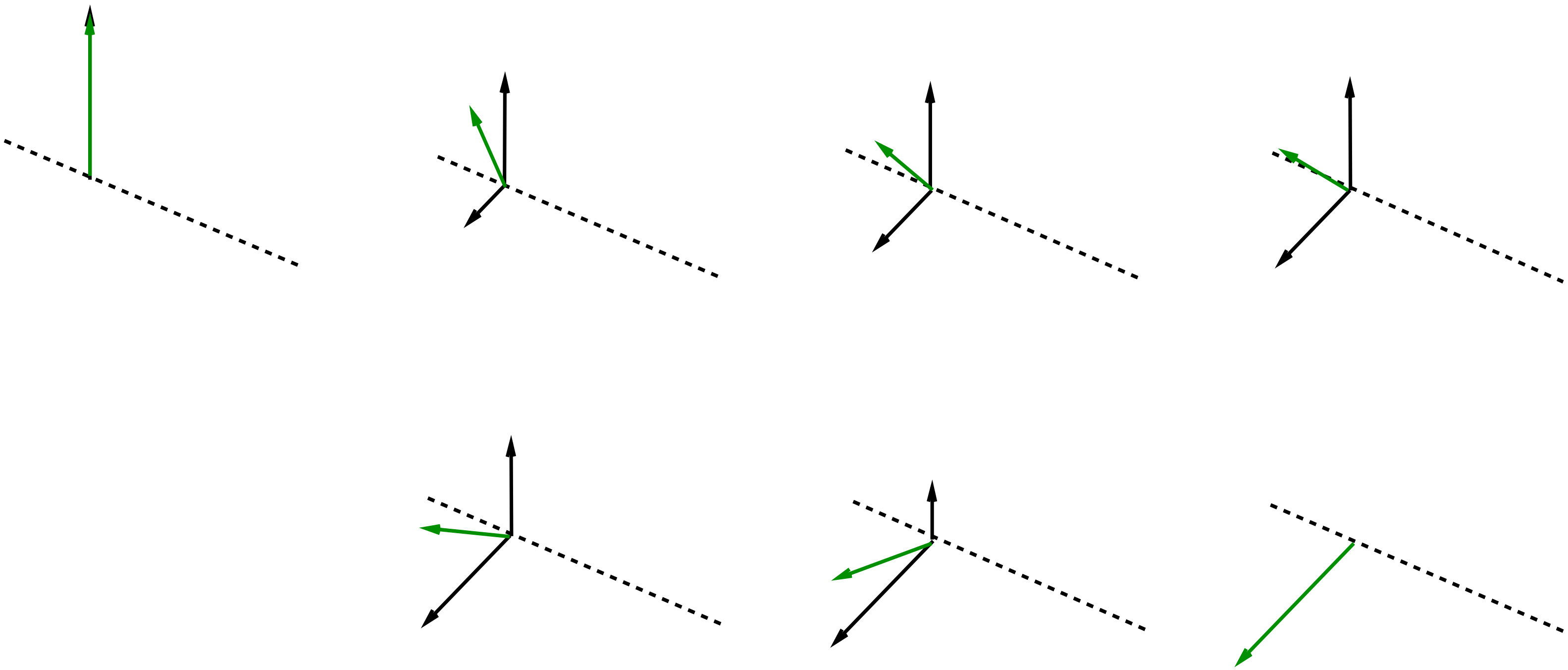


|B| constant: Decreasing B_z, Increasing B_y

$|\mathbf{B}| = 0.9$ T

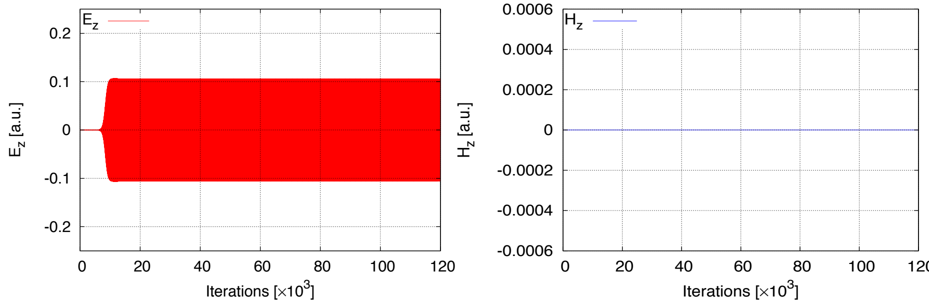
B_z = 0.9, 0.89, 0.85, 0.78, 0.67, 0.50, 0.0 T

B_y = 0.0, 0.15, 0.30, 0.45, 0.60, 0.75, 0.9 T

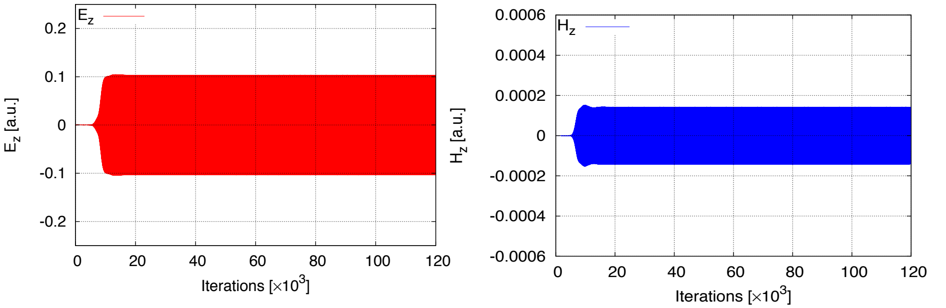


O-mode excitation E_z and x-mode coupled H_z signals

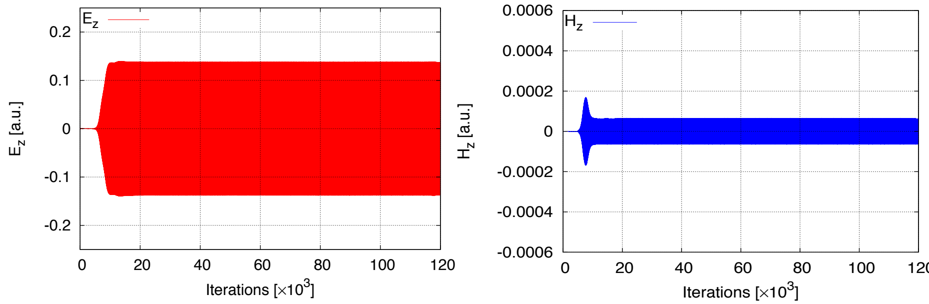
|**B**|=0.9T, B_z=0.9T, B_y=0T



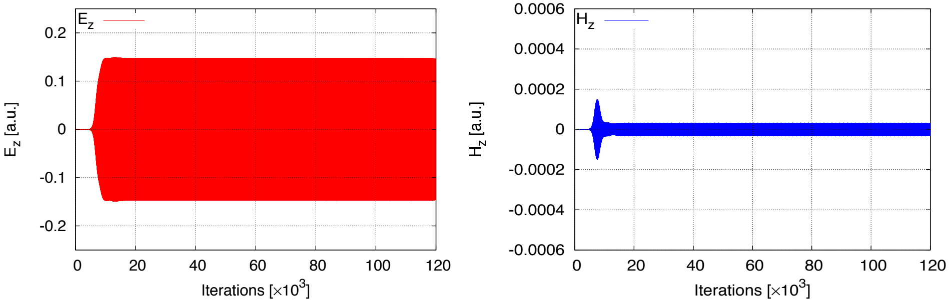
|**B**|=0.9T, B_z=0.85T, B_y=0.3T



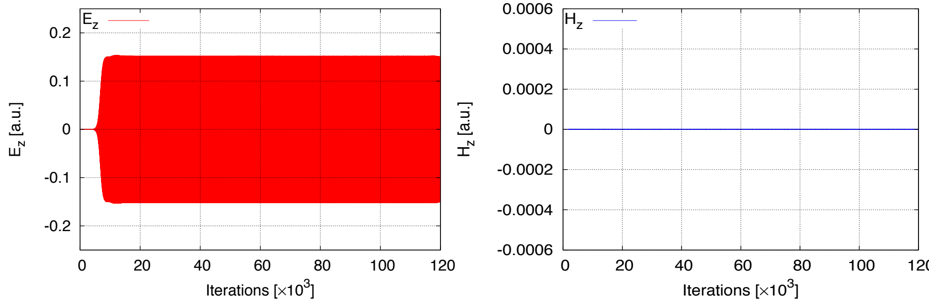
|**B**|=0.9T, B_z=0.67T, B_y=0.6T



|**B**|=0.9T, B_z=0.5T, B_y=0.75T

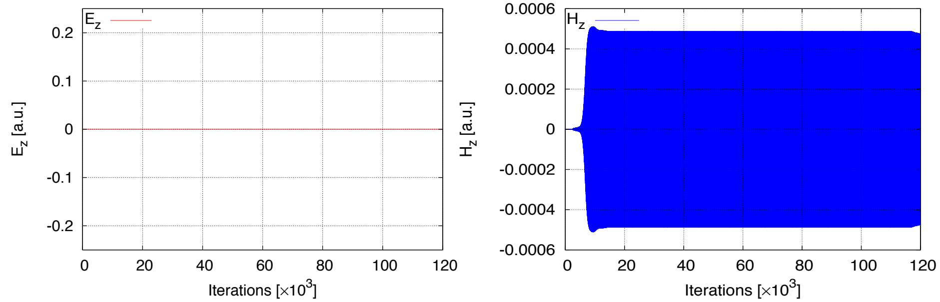


|**B**|=0.9T, B_z=0.0T, B_y=0.9T

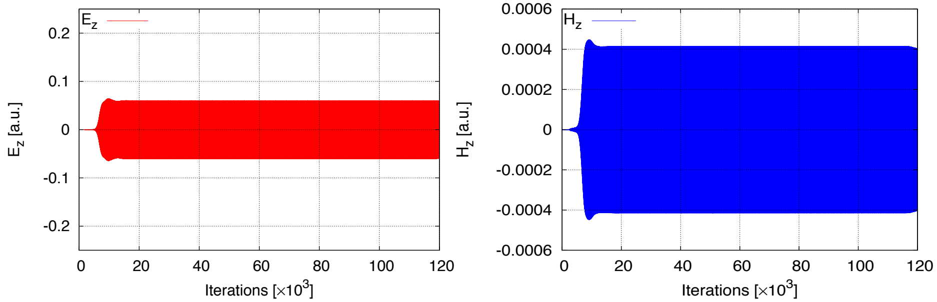


X-mode excitation H_z and o-mode coupled E_z signals

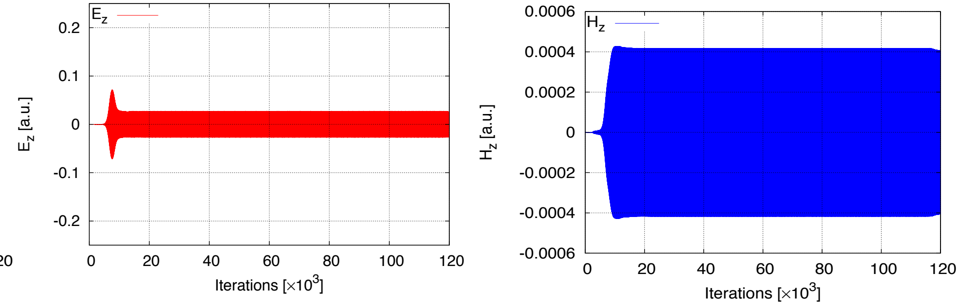
|**B**|=0.9T, B_z=0.9T, B_y=0T



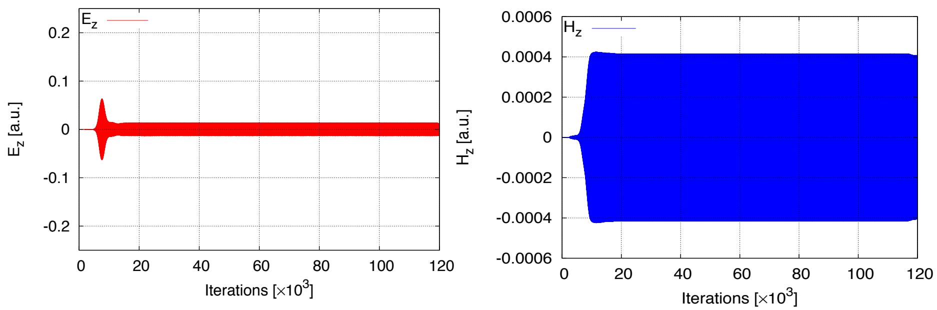
|**B**|=0.9T, B_z=0.85T, B_y=0.3T



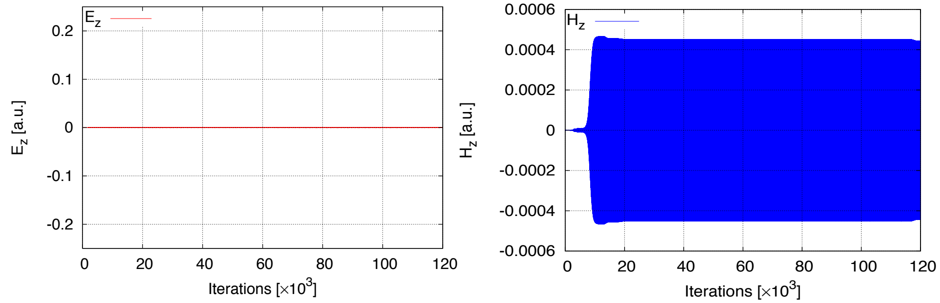
|**B**|=0.9T, B_z=0.67T, B_y=0.6T



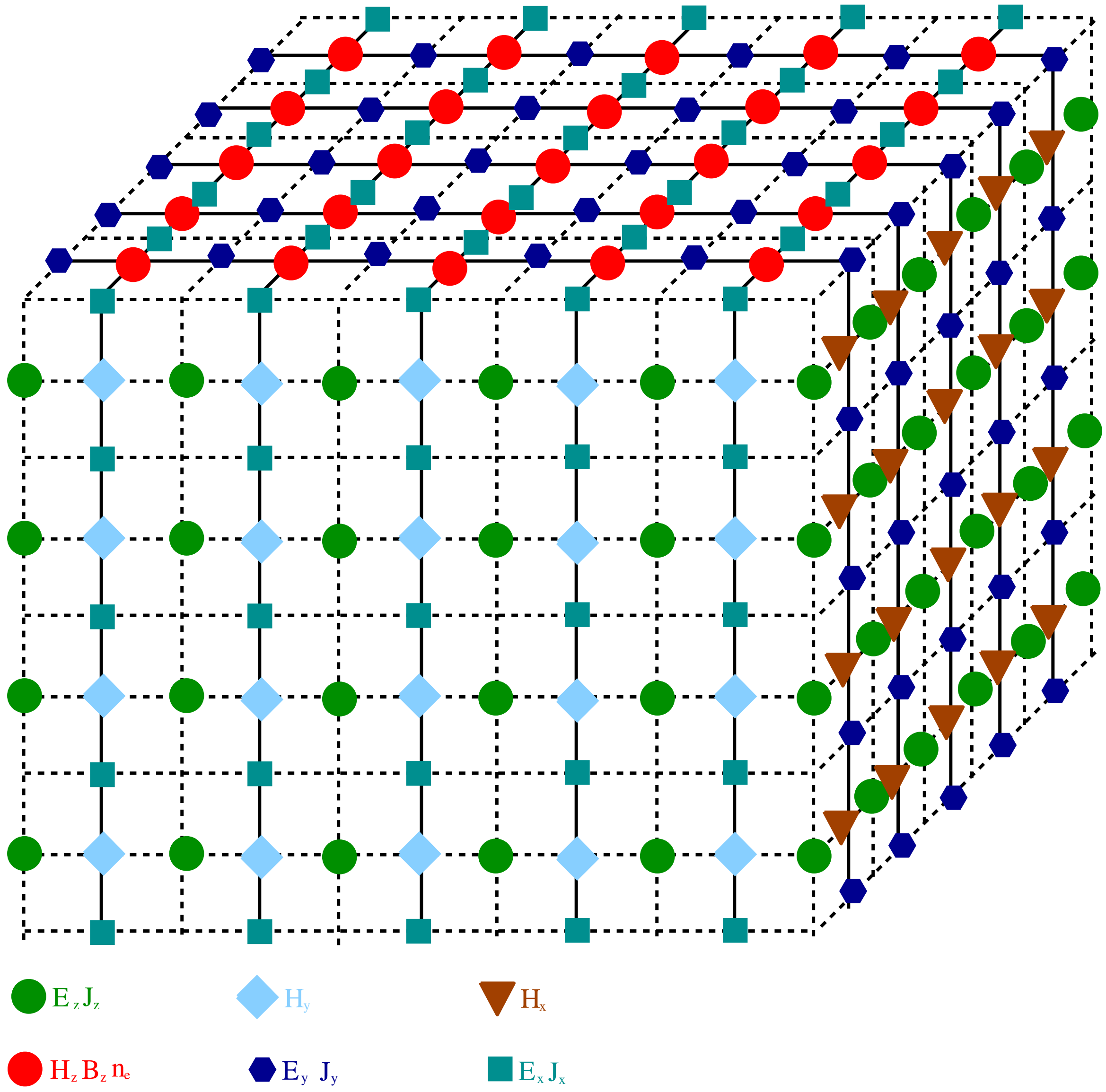
|**B**|=0.9T, B_z=0.5T, B_y=0.75T



|**B**|=0.9T, B_z=0.0T, B_y=0.9T



REFMULF not far from a 3D code...



New areas of application to our codes

From the usual treatment of Maxwell equations used in plasma physics

...and consequently reflected on our codes implementation

$$\begin{aligned} -\varepsilon_0 \partial_t \mathbf{E} + \nabla \times \mathbf{H} &= \mathbf{J} \\ \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} &= 0 \end{aligned}$$

Changing the vacuum permittivity $\boldsymbol{\varepsilon_0}$ to the absolute permittivity $\boldsymbol{\varepsilon=\varepsilon_0\varepsilon_r}$ permits the modeling of a **general isotropic linear lossless** dielectric

Including a conductivity $\boldsymbol{\sigma}$ through $\mathbf{J=\sigma E}$ allows the inclusion of **losses**

$$\begin{aligned} -\varepsilon \partial_t \mathbf{E} + \nabla \times \mathbf{H} &= \sigma \mathbf{E} \\ \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} &= 0 \end{aligned}$$

A new wide range of problems can be treated this way

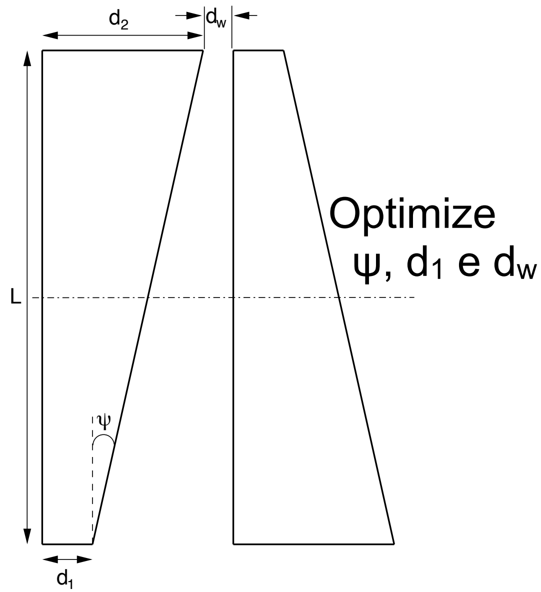
More complex modifications are possible...

Propagation in a dielectric medium (HDPE)

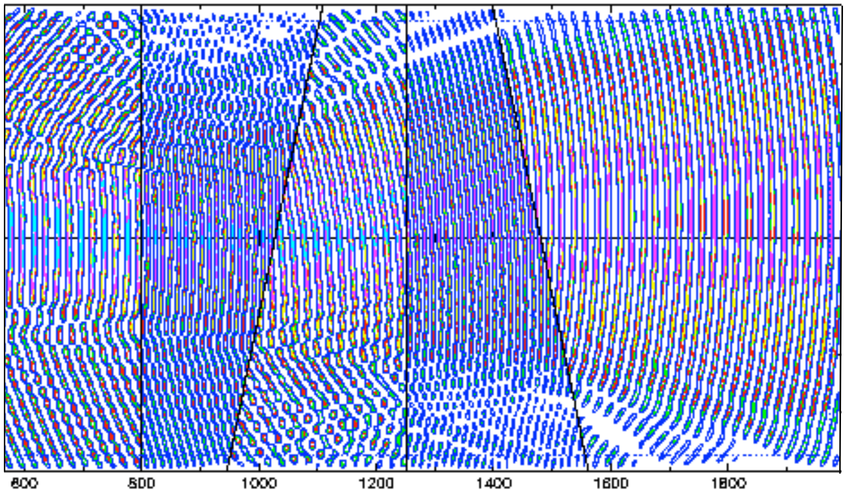
Propagation through an high-density polyethylene double wedge window **ε_r=2.3, tanδ=2.5×10^{-4}**

Propagation in a dielectric medium (HDPE)

Propagation through an high-density polyethylene double wedge window **ε_r=2.3, tanδ=2.5×10^{-4}**

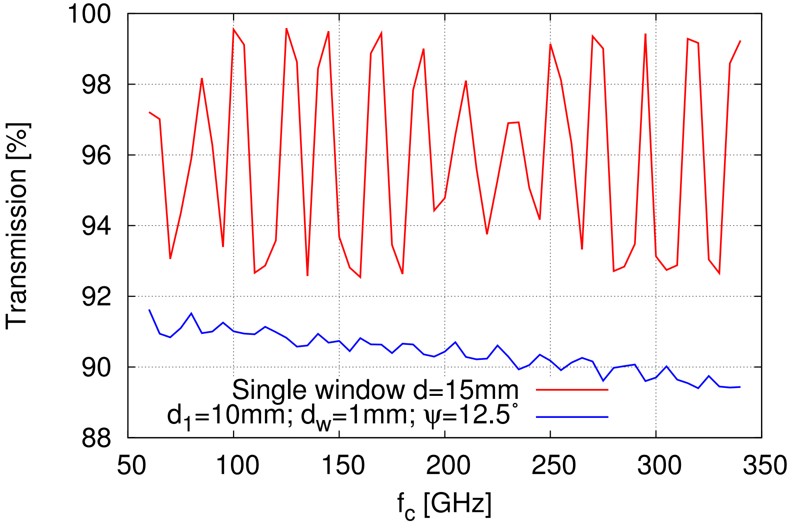


Snapshot of electric field



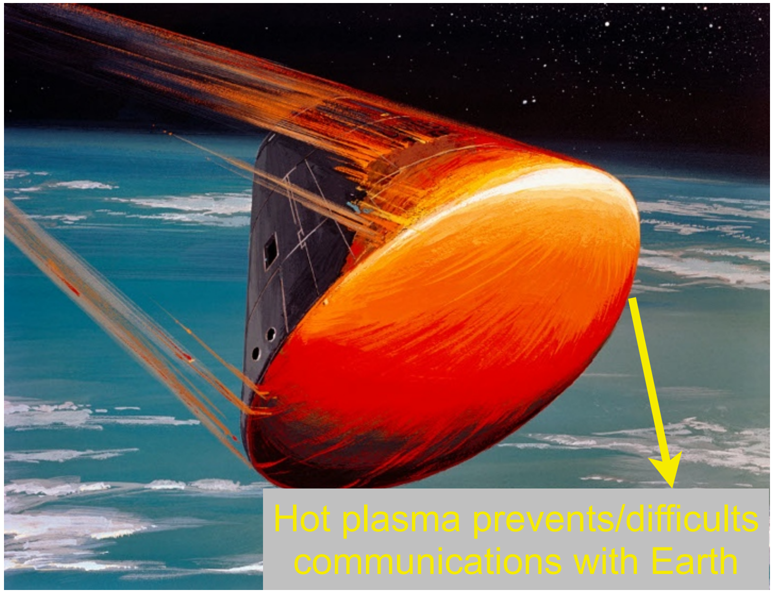
Filipe da Silva, Lisboa, 12/12/2014, GTM Seminar

Transmission coefficient



Mitigation of RF blackout for re-entry vehicles

ESA invitation to Tender (ITT) AO/1-7941/14/NL/MH



Need to transmit data during descent leads to examine mitigation techniques

- *Antenna placement*
- *Flow movement*
- *Cooling*
- *Promoting recombination*
- *Absorption*
- *Magnetic window*
- *Beam steering*

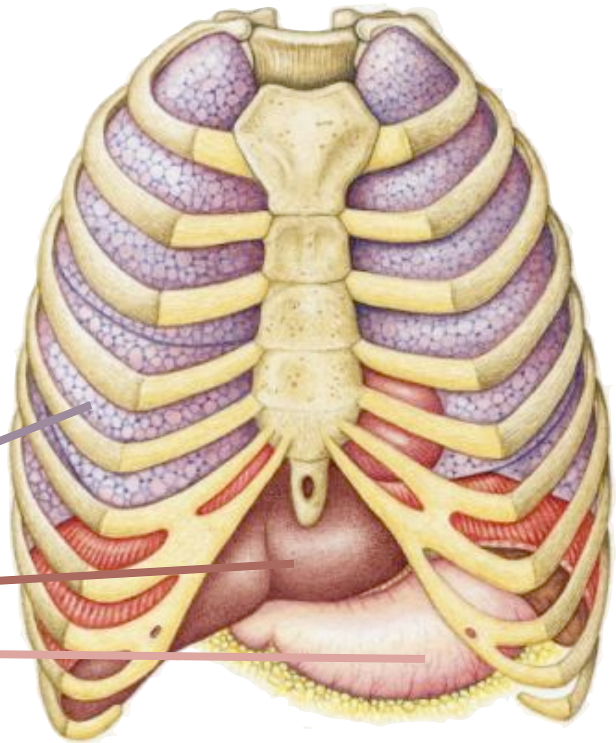
FDTD codes will be used to assess data transmission feasibility on the possible mitigation scenarios

Application to medicine and biology

How electromagnetic waves interact with biological tissue in general and with the human body in particular

- EM waves used to diagnose and/or to treat
- Effects of exposure of the human body to radiation
- Consequences of EM exposure in the short and long runs

Different tissues have distinct dielectric properties and which can be modeled with our FDTD codes with a region dependent relative permittivity $\varepsilon_r(\mathbf{r})$ and conductivity $\sigma(\mathbf{r})$



For 40 GHz:

Lungs inflated ε_r=46.118 σ=0.27253 S/m
Liver ε_r=98.419 σ=0.41054 S/m
Stomach ε_r=99.854 σ=0.85592 S/m
source: Instituto di Fisica Applicata Nello Carrara

We are committed to follow this topic since it presents an exciting field of opportunities involve several areas of knowledge such as numerical simulation and computational science, microwave technology or signal processing, IPFN Core Areas

Backup slides

- Radar e reflectometria
- Numerical dispersion
- Numerical stability
- (O)UTS
- GEMR (simplified)
- Correction to J PDE with plasma movement
- FDTD expressions

Usos do Radar

Os sistemas de radar são usados actualmente em variados contextos, tais como:

- Aplicações militares.
- Controlo do tráfego aéreo.
- Radio-astronomia.
- Radares náuticos.
- Sistemas da anticolisão em aeronaves.
- Vigilância do espaço exterior.
- Monitorização meteorológica.
- Observação geológica.
- Sistemas vigilância oceânica.
- (...)

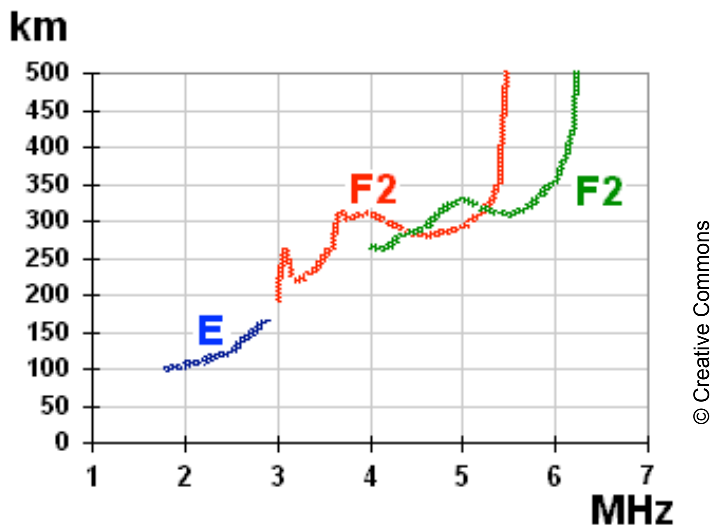
- **Estudos ionosféricos.**

Estudos ionosféricos

A **ionósfera** é a região exterior da atmosfera terrestre onde há uma elevada concentração de electrões livres e iões, capazes de afectar as ondas de rádio.

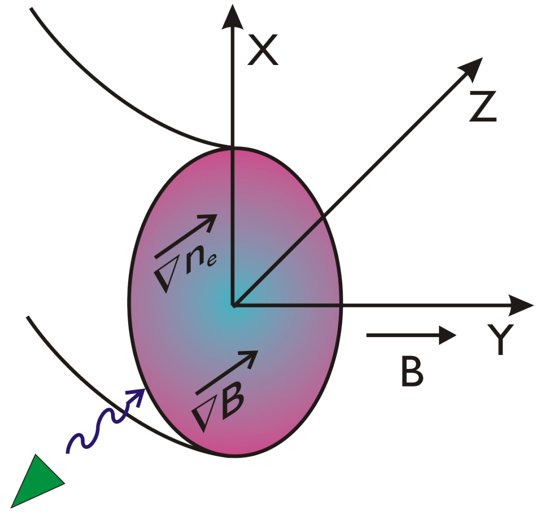
No seu estudo usa-se um tipo de radar denominado **ionosonda.**

- Sinal de sondagem enviado verticalmente.
- Varre uma gama de frequências de ~ 0,1 a 30 MHz.
- Sinal propaga-se até ser reflectido.
- Traçados da altura da reflexão versus frequências críticas - **Ionogramas.**



Reflectometria, *spin-off* dos estudos

- A reflectometria baseia-se nas técnicas de radar ionosférico.
- Sinal lançado perpendicularmente às iso-superfícies de índice de densidade do plasma.
- Propaga-se até parte real do índice de refracção se anular (corte) sendo reflectida até ao ponto de partida.



- Sinal pode retornar à antena por retrodifusão de Bragg, sem que haja reflexão.
- Reflectometria Doppler explora retrodifusão de Bragg injectando a onda num ângulo oblíquo.

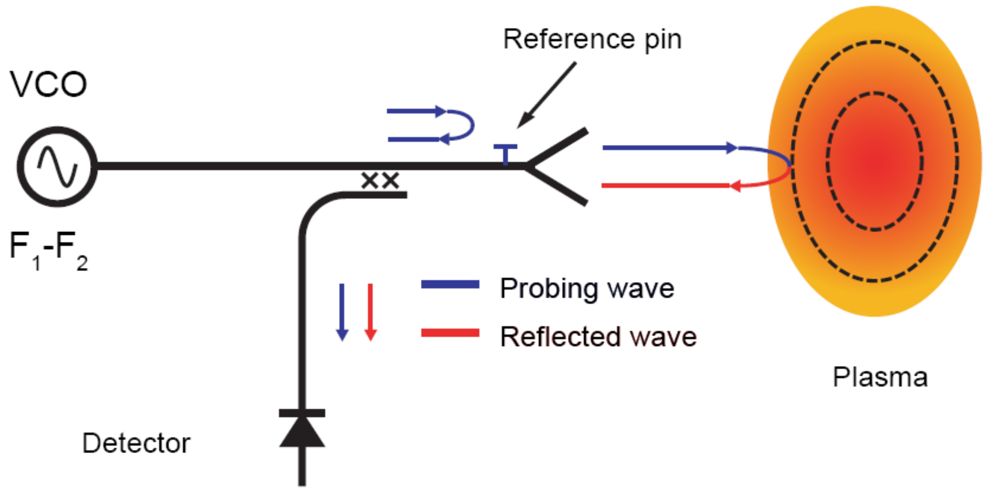
Esquema de um reflectómetro

- Sinal enviado ao plasma

$$s_{src}(t) = A_{src}\cos(\omega t)$$

- Propaga-se sendo reflectido na posição de corte r_c

$$s_{rcv}(t) = A_{ant}\cos(\omega t + \varphi)$$



$$\varphi = \varphi_0 + \varphi_\mu + \varphi_p$$

- A onda reflectida apresenta um desfasamento φ devido ao circuíto de micro-ondas φ_μ, às propagações no vácuo φ_0 e no **plasma** $\boldsymbol{\varphi_p}$.

$$\varphi_p = \frac{4\pi f}{c}\int_0^{r_c} N(r)\mathrm{d}r - \varphi_k$$

- A fase $\boldsymbol{\varphi_p}$ traduz a propagação da onda ao longo de um percurso descrito por um índice de refração *N(r)*, contendo informação sobre a densidade n_e

- Modo-O $\varphi_p = \varphi_p[f, N_O(n_e)]$
- Modo-X $\varphi_p = \varphi_p[f, N_X(n_e, B_0)]$

Dispersion relation

- Variation of propagating wave's wavelength λ with frequency *f*

$$k(\omega), \qquad k = 2\pi\lambda^{-1}, \qquad \omega = 2\pi f$$

phasor form: $u(x,t) = \mathrm{e}^{j(\omega t - kx)}, \qquad j = \sqrt{-1}$

$$\frac{\partial^2 u}{\partial t^2} = \mathrm{c}^2 \frac{\partial^2 u}{\partial x^2} \Longrightarrow (j\omega)^2 \mathrm{e}^{j(\omega t - kx)} = \mathrm{c}^2 (-jk)^2 \mathrm{e}^{j(\omega t - kx)} \Longrightarrow \omega^2 = \mathrm{c}^2 k^2$$

dispersion relation $\quad k = \pm \frac{\omega}{c}$

phase velocity: $v_{ph} = \frac{\omega}{k} = \pm c$ group velocity: $v_{gr} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \pm c$

Numerical dispersion relation I

$$u(x,t) = \mathrm{e}^{j(\omega t - kx)} \quad \text{at} \quad (x_i, t_n)$$
$$\tilde{k} = \tilde{k}_{real} + j\tilde{k}_{imag}$$

$$u_i^n = \mathrm{e}^{j(\omega n\Delta t - \tilde{k} i \Delta x)} = \mathrm{e}^{\tilde{k}_{imag} i \Delta x}\mathrm{e}^{j(\omega n \Delta t - \tilde{k}_{real} i \Delta x)}$$

Introducing u_i^n in the expression for u_i^{n+1}

$$\frac{\mathrm{e}^{j\omega\Delta t} + \mathrm{e}^{-j\omega\Delta t}}{2} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(\frac{\mathrm{e}^{j\tilde{k}\Delta x} + \mathrm{e}^{-j\tilde{k}\Delta x}}{2} - 1\right) + 1$$

$$\cos(\omega \Delta t) = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left[\cos(\tilde{k}\Delta x) - 1\right] + 1$$

numerical dispersion relation:

$$\tilde{k} = \frac{1}{\Delta x}\cos^{-1}\left\{1 + \left(\frac{\Delta x}{c\Delta t}\right)^2 \left[\cos(\omega\Delta t) - 1\right]\right\}$$

Numerical dispersion relation II

Case 1: cΔt=Δx *(Magical step):*

$\tilde{k} = k$ free space wavenumber

"Magic" works just on 1D

Marginally stable

Case 2: Δt→0 and Δx→0 *(Very fine sampling in space and time):*

$$\tilde{k} \cong \frac{1}{\Delta x}\cos^{-1}\left[1 - \frac{(k\Delta x)^2}{2}\right]$$ (ωΔt→0)

$$\tilde{k} \cong \frac{1}{\Delta x}(k\Delta x)$$ (kΔx→0)

$\tilde{k} \cong k$ ≅free space wavenumber

Numerical dispersion relation III

Case 3: $c\Delta t = \Delta x/2 \qquad \tilde{k} = \frac{1}{\Delta x}\cos^{-1}\left\{1 + 4\left[\cos\left(\frac{k\Delta x}{2}\right) - 1\right]\right\}$

$$\Delta x = \lambda_0/10 \qquad \tilde{k} = \frac{1}{\Delta x}\cos^{-1}(0.802) = \frac{0.63642}{\Delta x}$$

$$\tilde{v}_{ph} = \frac{\omega}{\tilde{k}} = 0.9873\text{c} \qquad \epsilon = 1.27\% \quad \text{less than c}$$

$$10\lambda_0(100 \text{ space cells}) \longrightarrow \left[\frac{100 - 98.73}{10}\right] \cdot 360^\circ = 45.72^\circ$$

$$\Delta x = \lambda_0/20 \qquad \tilde{k} = \frac{1}{\Delta x}\cos^{-1}(0.9508) = \frac{0.31514}{\Delta x}$$

$$\tilde{v}_{ph} = \frac{\omega}{\tilde{k}} = 0.99689\text{c} \qquad \epsilon = 0.31\% \quad \text{less than c}$$

$$10\lambda_0(200 \text{ space cells}) \longrightarrow \left[\frac{200 - 199.378}{20}\right] \cdot 360^\circ = 11.20^\circ$$

$$O\left[(\Delta x)^2\right]$$

Numerical stability

Numerical instability is a possibility with explicit numerical differential equation solvers

- Spectral technique analysis (von Neumann)
  - Error at any point expressed as a finite spatial Fourier series
  - Each Fourier term with unity-or-less growth factor over one time-step
  - If every term is initially bounded, each term remains bounded
- Complex-frequency analysis of numerical dispersion

$$\tilde{k} = \tilde{k}_{real} + j\tilde{k}_{imag}$$

$$\tilde{\omega} = \tilde{\omega}_{real} + j\tilde{\omega}_{imag}$$

Complex-frequency analysis of numerical dispersion

$$\tilde{k} = \tilde{k}_{real} + j\tilde{k}_{imag} \qquad \tilde{\omega} = \tilde{\omega}_{real} + j\tilde{\omega}_{imag}$$

$$u_i^n = \mathrm{e}^{j(\tilde{\omega} n\Delta t - \tilde{k} i \Delta x)} = \mathrm{e}^{-\tilde{\omega}_{imag} n\Delta t}\mathrm{e}^{j(\tilde{\omega}_{real} n\Delta t - \tilde{k} i \Delta x)}$$

dispersion relation: $$\cos(\tilde{\omega}\Delta t) = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left[\cos(\tilde{k}\Delta x) - 1\right] + 1$$

$S = \mathrm{c}\Delta t/\Delta x$ (Courant stability factor)

$$\tilde{\omega} = \frac{1}{\Delta t}\left[\frac{\pi}{2} - \sin^{-1}(\xi)\right] \quad \text{with} \quad \xi = S^2\left[\cos(\tilde{k}\Delta x) - 1\right] + 1$$

$$1 - 2S^2 \leq \xi \leq 1 \qquad \forall \tilde{k} \in \mathbb{R}$$

Courant stability factor S

$$(a) -1 \leq \xi \leq 1 \quad (0 \leq S \leq 1)$$

$$\sin^{-1}(\xi) \in \mathbb{R} \qquad \tilde{\omega} \in \mathbb{R} \Longleftrightarrow \tilde{\omega}_{imag} = 0$$

$$u_i^n = \mathrm{e}^{j(\tilde{\omega}_{real} n\Delta t - \tilde{k} i \Delta x)}$$

$$(b) 1 - 2S^2 \leq \xi \leq -1 \quad (S > 1)$$

$$\sin^{-1}(\xi) \in \mathbb{C}, \qquad \sin^{-1}(\xi) = -j \ln\left(j\xi + \sqrt{1-\xi^2}\right)$$

$$\tilde{\omega} = \frac{1}{\Delta t}\left[\frac{\pi}{2} + j\ln\left(j\xi + j\sqrt{\xi^2 - 1}\right)\right] = \frac{1}{\Delta t}\left[\pi + j\ln\left(-\xi - \sqrt{\xi^2-1}\right)\right]$$

$$\tilde{\omega}_{real} = \frac{\pi}{\Delta t} \qquad \tilde{\omega}_{imag} = \frac{1}{\Delta t}\ln\left(-\xi - \sqrt{\xi^2 - 1}\right)$$

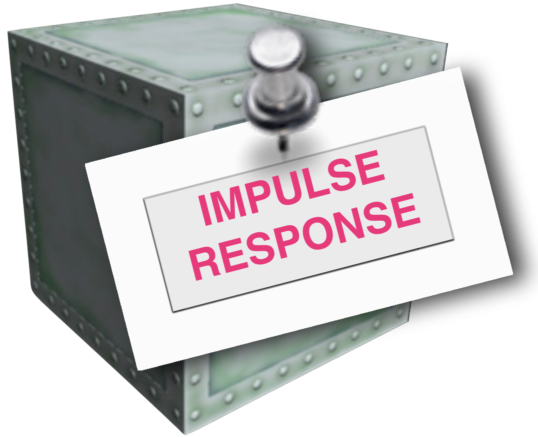
$$u_i^n = \left(\frac{1}{-\xi - \sqrt{\xi^2 - 1}}\right)^{nth\,\text{power}} \mathrm{e}^{j\left[(\pi/\Delta t)n\Delta t - \tilde{k} i \Delta x\right]}$$

Linear Time Invariant Systems and Impulse Response

A system T $\mathcal{T} : x \longmapsto y$ **is:**

Linear $\mathcal{T}\{ax_1 + bx_2\} = a\mathcal{T}\{x_1\} + b\mathcal{T}\{x_2\}$

Time invariant $\mathcal{T}\{x(t - t_0)\} = (\mathcal{T}\{x\})(t - t_0)$



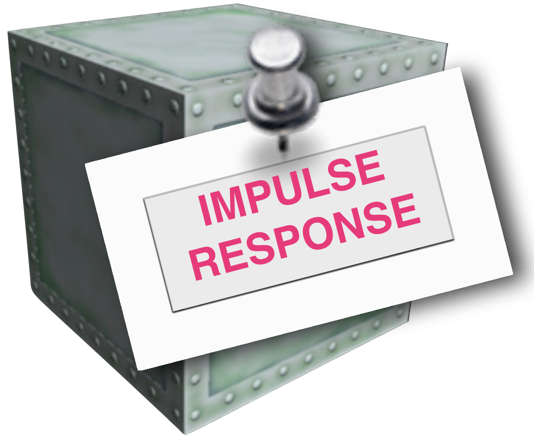
Impulse response $h(t) = \mathcal{T}\{\delta(t)\}$

In a discrete system $h[n] = \mathcal{T}\{\delta[n]\}$

An input $x[n] = f(n\Delta t)$ **is a linear combination of weighted, time translated** $\delta[n]$

$$\mathcal{T} : \quad x[n] = \sum_{k=-\infty}^{+\infty} f(n\Delta t)\delta[n-k] \longmapsto y[n] = \sum_{k=-\infty}^{+\infty} f(n\Delta t)h[n-k]$$

Linear Time Invariant Systems and Impulse Response



A system T $\mathcal{T} : x \longmapsto y$ **is:**

Linear $\mathcal{T}\{ax_1 + bx_2\} = a\mathcal{T}\{x_1\} + b\mathcal{T}\{x_2\}$

Time invariant $\mathcal{T}\{x(t-t_0)\} = (\mathcal{T}\{x\})\,(t-t_0)$

Impulse response $h(t) = \mathcal{T}\{\delta(t)\}$

In a discrete system $h[n] = \mathcal{T}\{\delta[n]\}$

An input $x[n] = f(n\Delta t)$ **is a linear combination of weighted, time translated** $\delta[n]$

$$\mathcal{T} : \quad x[n] = \sum_{k=-\infty}^{+\infty} f(n\Delta t)\delta[n-k] \longmapsto y[n] = \sum_{k=-\infty}^{+\infty} f(n\Delta t)h[n-k]$$

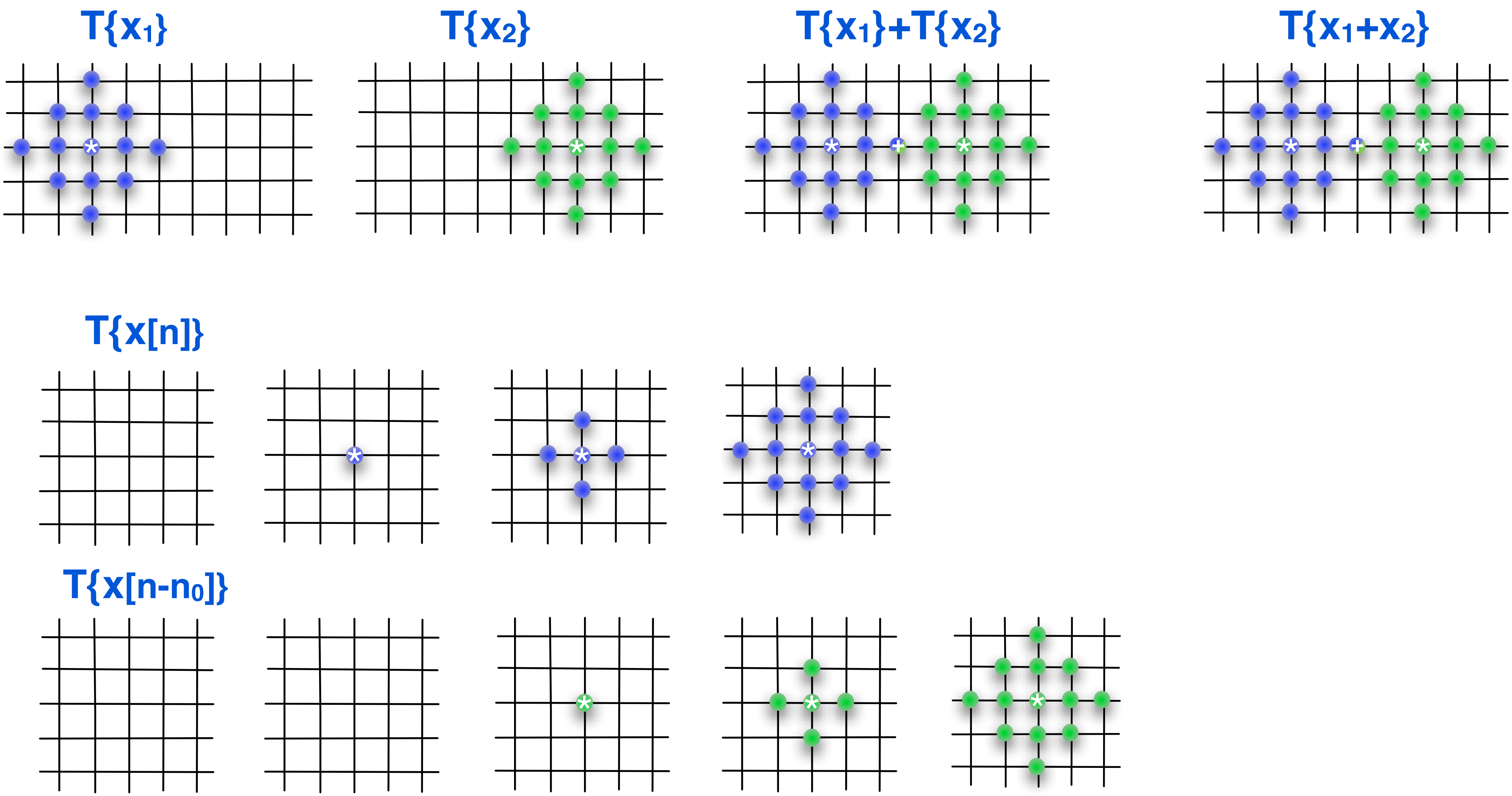
For a causal system (k=0)

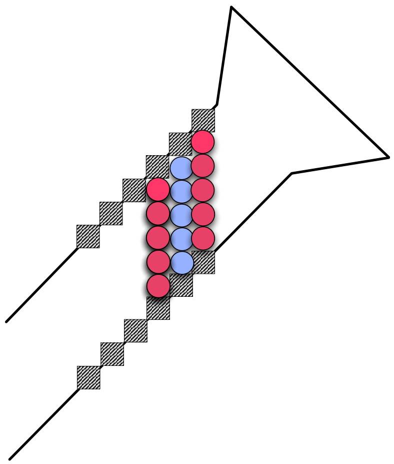
FIR (N-1)

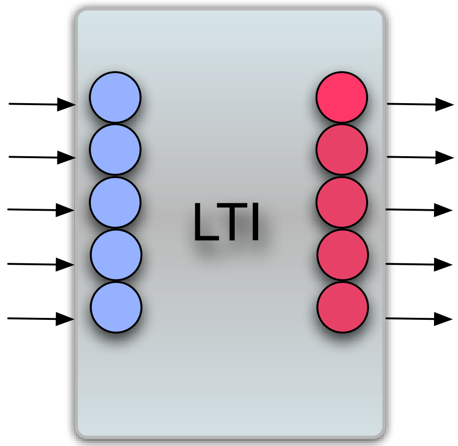
IIR (ϵ≠0)

$$y[n] = \sum_{k=0}^{N-1} f(n\Delta t)h[n-k] + \epsilon$$

FDTD schema is a LTI





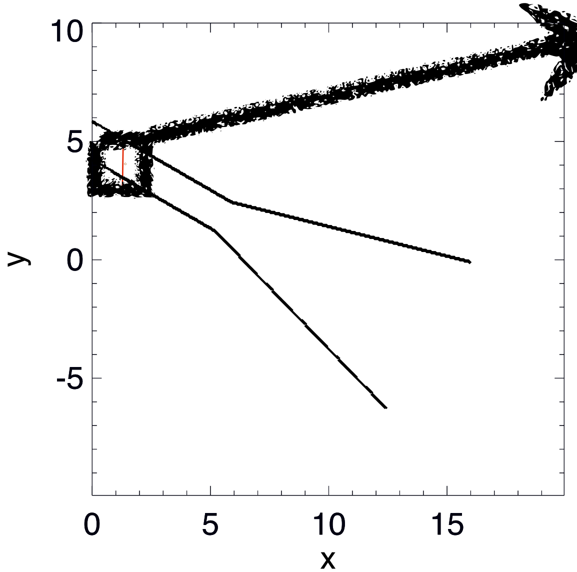


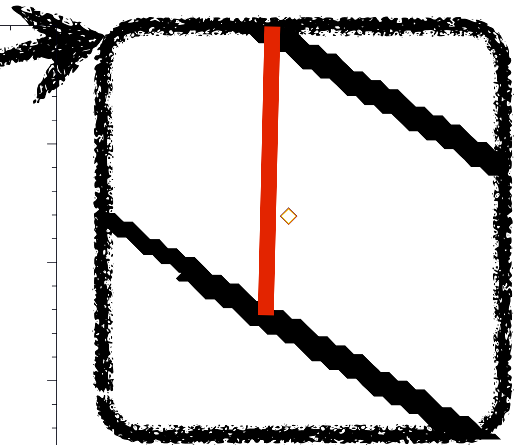
M input N output system

For each output N

$$y_N[n] = \sum_M y_{MN}[n] = \sum_M \sum_k x_M[n] h_{MN}[n-k]$$

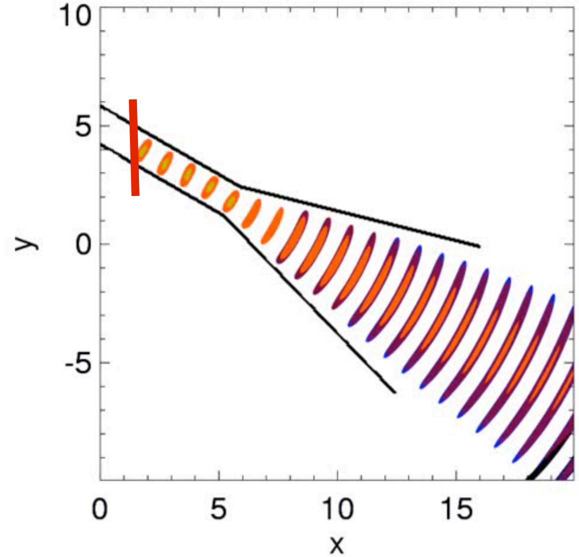
Implementation example



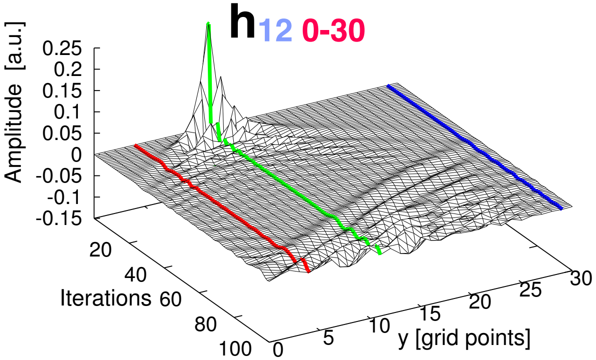


Impulse at "12"

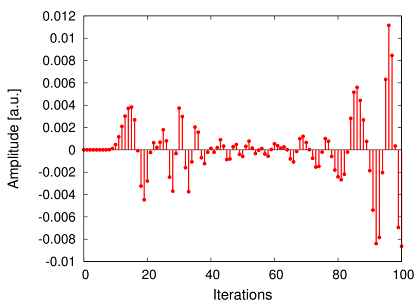
Oblique UTS



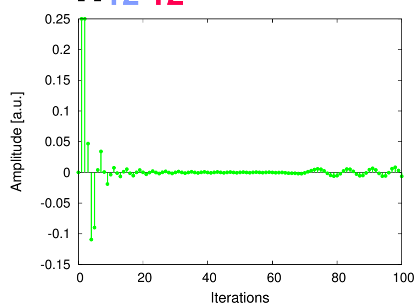
h_{12} 0-30



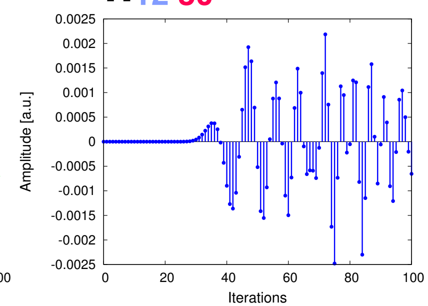
h_{12} 5



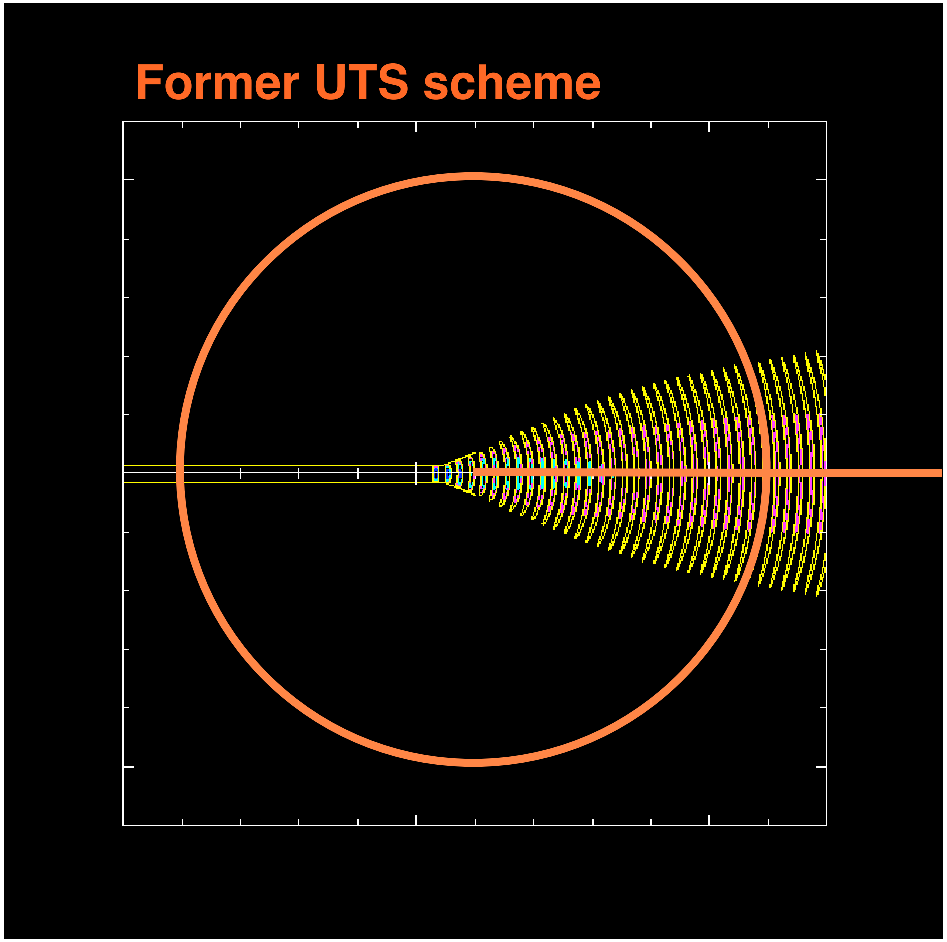
h_{12} 12



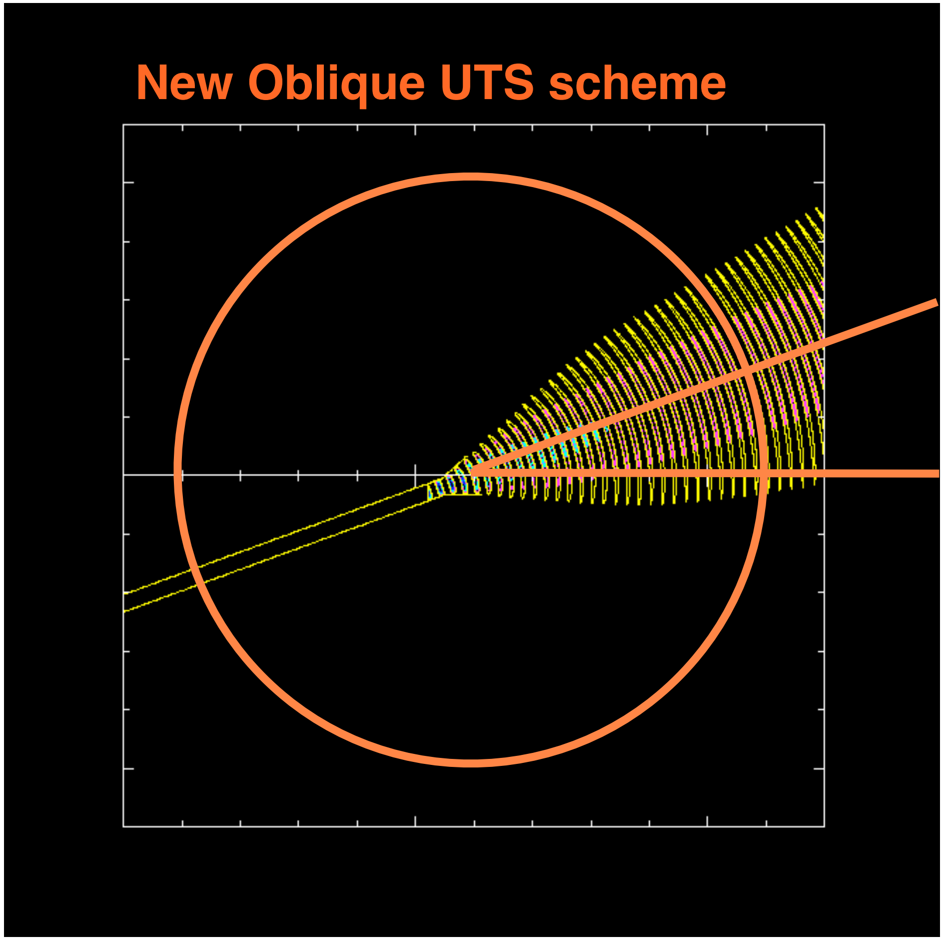
h_{12} 30



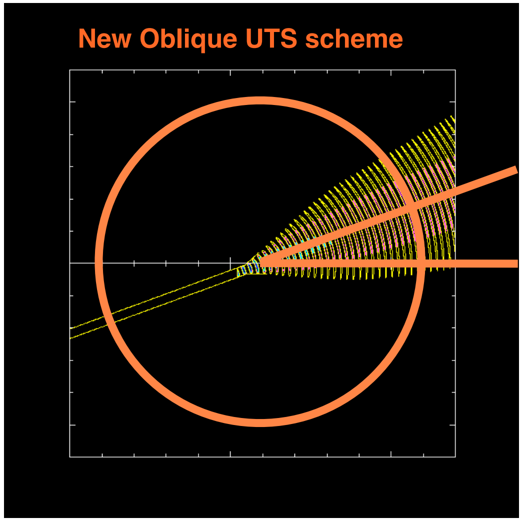
Radiation pattern behavior

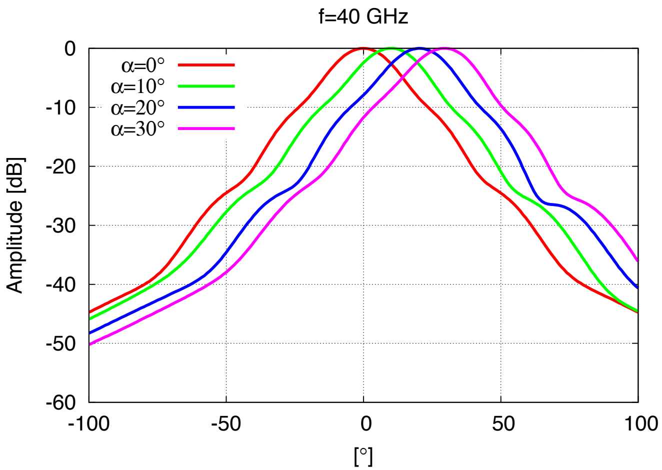


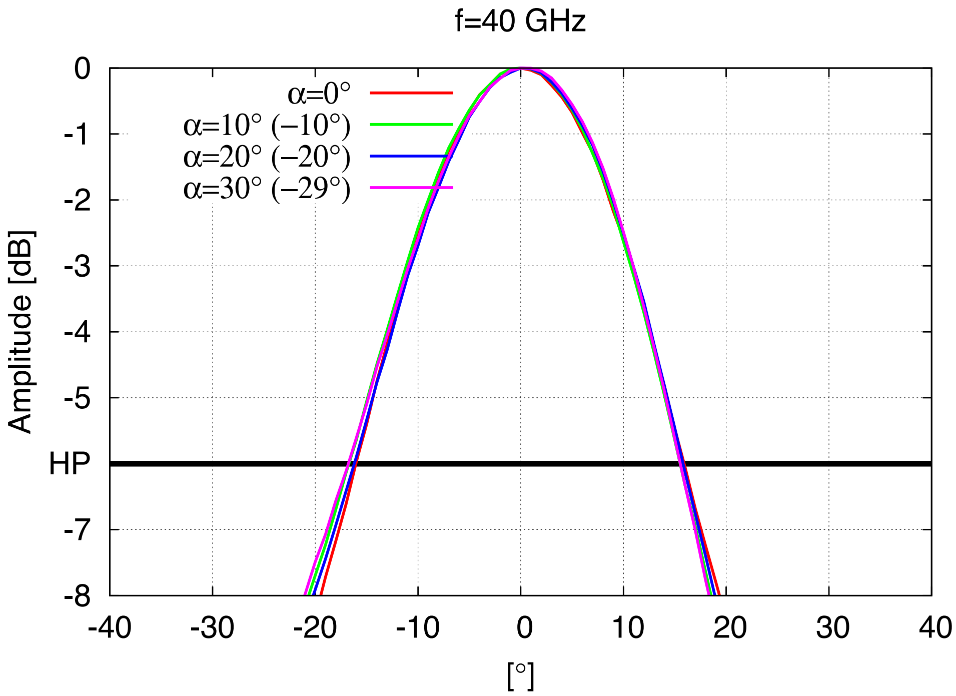
Radiation pattern behavior



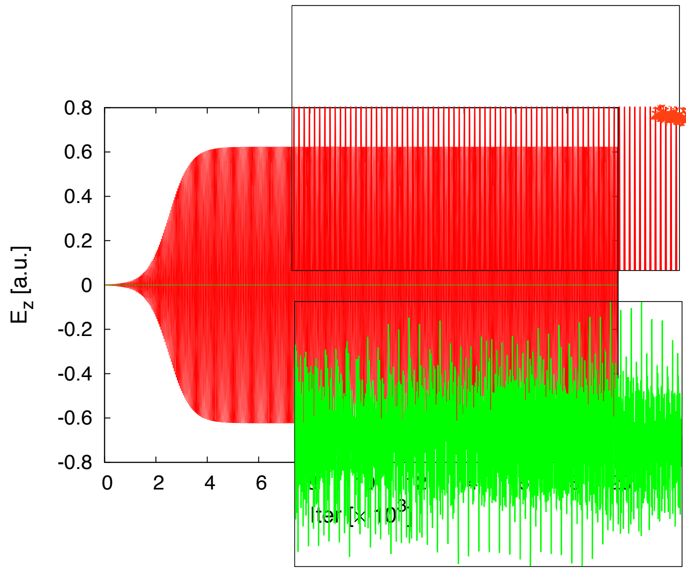
Radiation pattern behavior

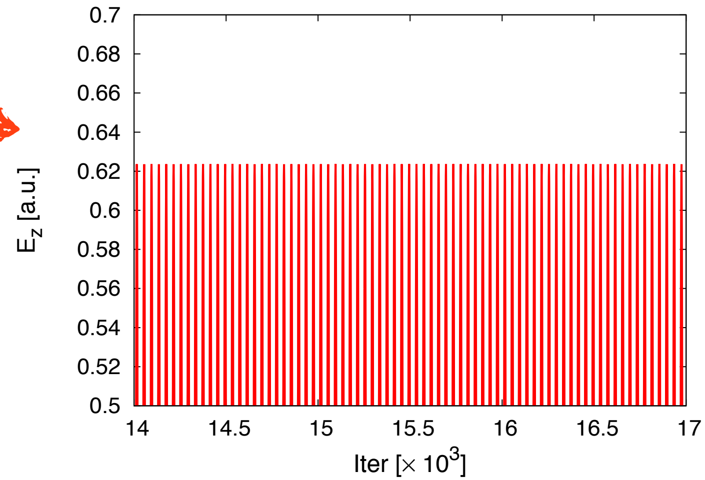




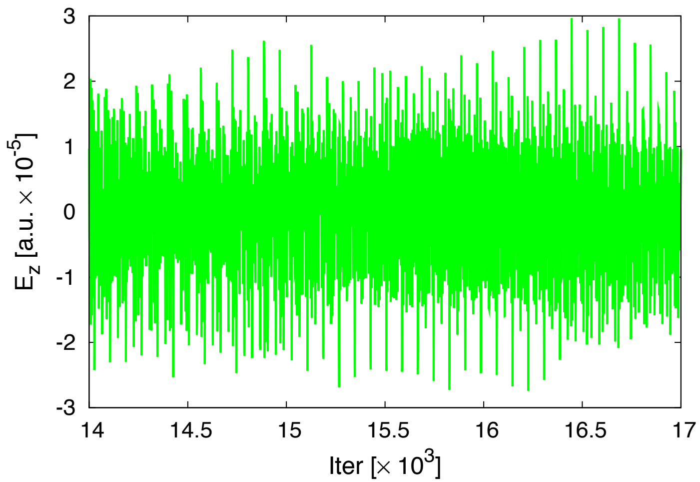


Level of correction of the OUTS





- In **RED** the corrected emission
- In **GREEN** the correction residue
  On numeric noise level



☑ **Correction ratio ~25,000 (44 dB)**

GEMR (simplified) model equations and numerics

Isothermal equations for illustrations purposes.
Actual model evolves temperature dynamics.

Continuity

$$\frac{\partial \tilde{n}_z}{\partial t} = -\tilde{v}_{zE}\cdot\nabla\tilde{n}_z - B\nabla_{\|}\frac{\tilde{v}_{z\|}}{B} + \mathcal{K}(\tilde{\varphi}_z + \tau_z\tilde{n}_z)$$

$\partial_t\tilde{n}_z - \omega_n\partial_y\tilde{\varphi}_z - [\tilde{\varphi}_z,\tilde{n}_z]$ $\quad B\partial_s\tilde{v}_{z\|}/B + \hat{\beta}[\tilde{A}_{\|},\tilde{v}_{z\|}]$

$$\frac{\partial F(f)}{\partial t}$$

$$S(f)$$

Parallel force balance

$$\frac{\partial}{\partial t}(\hat{\beta}\tilde{A}_{\|} - \hat{\mu}_z\tilde{v}_{z\|}) = \hat{\mu}_z v_{zE}\cdot\nabla v_{z\|} - \nabla_{\|}(\tilde{\varphi}_z + \tau_z\tilde{n}_z) + \hat{\mu}_z\mathcal{K}(2\tilde{v}_{z\|}) - C\tilde{J}_{\|}$$

Induction

$$-\nabla_\perp^2\tilde{A}_{\|} = \tilde{J}_{\|} = \tilde{u}_{\|} - \tilde{v}_{\|}$$

Polarisation

$$\Gamma_0^{1/2}\tilde{n}_{\mathrm{i}} + \frac{\Gamma_0 - 1}{\tau_{\mathrm{i}}}\tilde{\varphi} = \tilde{n}_{\mathrm{e}}$$

$$(\alpha^2 - \nabla_\perp^2)f = F$$

GEMR (simplified) model equations and numerics

Numerical scheme $\dfrac{\partial F(f)}{\partial t} = S(f) + D(f)$ and $(\alpha^2 - \nabla_\perp^2)f = F$

- $S(f)$ terms - spatial discretisation
  - parallel derivatives (& curvature): 2nd order centered finite differences
  - Poisson brackets - nonlinearities: 2nd or 4th order finite differences Arakawa scheme

- $\dfrac{\partial F(f)}{\partial t}$ terms - time advance: 3rd order time-accurate Karniadakis scheme

$$F_{n+1} = \frac{6}{11}\left[3F_n - \frac{3}{2}F_{n-1} + \frac{1}{3}F_{n-2} + \Delta t(3S_n - 3S_{n-1} + S_{n-2})\right]$$

- $D(f)$ terms - sub-grid dissipation: viscosity and hyper-viscosity

$$F_{n+1} \leftarrow F_{n+1} + \Delta t D_n \qquad \text{with} \qquad D_n = -\nu_\perp \nabla_\perp^4 f_n - \nu_\parallel \nabla_\parallel^2 f_n$$

- $f_{n+1} \leftarrow F_{n+1}$ recover fluid variables: Helmholtz solve for the fields

Parallelisation: MPI domain decomposition in parallel and radial directions

Correction to J PDE with plasma movement

$$m_e \partial_t u_e = eE + e u_e \times B_0$$

$$m_e \partial_t \left(\frac{J}{n_e e}\right) = eE + \frac{J \times B_0}{n_e}$$

$$m_e \left[\frac{1}{n_e e} \partial_t J + J \partial_t \left(\frac{1}{e n_e}\right)\right] = eE - \frac{B_0 \times J}{n_e}$$

$$\partial_t J = \varepsilon_0 \omega_p^2 E + \omega_c b \times J - J \partial_t \left(\frac{1}{e n_e}\right) n_e e$$

$$\partial_t J = \varepsilon_0 \omega_p^2 E + \omega_c b \times J - J \left(-\frac{1}{n_e^2}\right) \partial_t n_e \, n_e$$

Sequência de cálculo

Kernel (M) Xu & Yuan

$$H_z^{n+1/2}$$

$$J_x^{n+1/2}$$

$$J_y^{n+1/2}$$

$$E_x^{n+1}$$

$$E_y^{n+1}$$

Kernel Déspres & Pinto

$$H_z^{n+1/2}$$

$$J_x^{n+1}$$

$$J_y^{n+1}$$

$$E_x^{n+1}$$

$$E_y^{n+1}$$

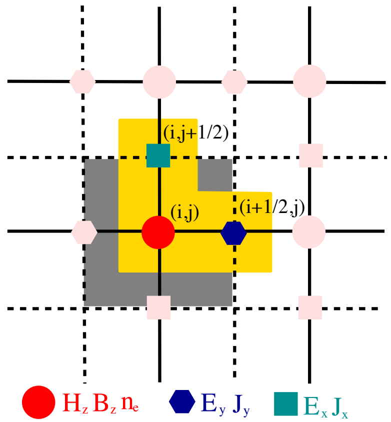
Cálculo de H_z (M Xu & Yuan)

calcHzFieldKXY

$$H_{zx}^{n+1/2}(i,j) = \left(\frac{1-\frac{\Delta t\sigma_x^\star(i,j)}{2\mu_0}}{1+\frac{\Delta t\sigma_x^\star(i,j)}{2\mu_0}}\right)H_{zx}^{n-1/2}(i,j) - \frac{\Delta t}{\mu_0\Delta x\left(1+\frac{\Delta t\sigma_x^\star(i,j)}{2\mu_0}\right)}\left[E_y^n(i+1/2,j) - E_y^n(i-1/2,j)\right]$$

$$H_{zy}^{n+1/2}(i,j) = \left(\frac{1-\frac{\Delta t\sigma_y^\star(i,j)}{2\mu_0}}{1+\frac{\Delta t\sigma_y^\star(i,j)}{2\mu_0}}\right)H_{zy}^{n-1/2}(i,j) + \frac{\Delta t}{\mu_0\Delta y\left(1+\frac{\Delta t\sigma_y^\star(i,j)}{2\mu_0}\right)}\left[E_x^n(i,j+1/2) - E_x^n(i,j-1/2)\right]$$

Cálculo da densidade de corrente $J_{x,y}$ (M Xu & Yuan)



calcJxyFieldKXY

$$\omega_c(i,j) = \frac{\mathrm{e}}{\mathrm{m_e}} B_z(i,j) \qquad \omega_p^2(i,j+1/2) = \frac{\mathrm{e}^2}{\mathrm{m_e}\varepsilon_0} n_e(i,j)$$

$$\begin{aligned} J_x^{n+1/2}(i,j+1/2) &= \frac{1-\frac{\omega_c^2\Delta t^2}{4}}{1+\frac{\omega_c^2\Delta t^2}{4}} J_x^{n-1/2}(i,j+1/2) - \frac{\omega_c \Delta t}{1+\frac{\omega_c^2\Delta t^2}{4}} J_y^{n-1/2}(i+1/2,j) \\ &+ \frac{\varepsilon_0 \omega_p^2 \Delta t}{1+\frac{\omega_c^2\Delta t^2}{4}} \left(E_x^n(i,j+1/2) - \frac{\omega_c \Delta t}{2} E_y^n(i+1/2,j) \right) \end{aligned}$$

$$\begin{aligned} J_y^{n+1/2}(i+1/2,j) &= \frac{1-\frac{\omega_c^2\Delta t^2}{4}}{1+\frac{\omega_c^2\Delta t^2}{4}} J_y^{n-1/2}(i+1/2,j) + \frac{\omega_c \Delta t}{1+\frac{\omega_c^2\Delta t^2}{4}} J_x^{n-1/2}(i,j+1/2) \\ &+ \frac{\varepsilon_0 \omega_p^2 \Delta t}{1+\frac{\omega_c^2\Delta t^2}{4}} \left(E_y^n(i+1/2,j) + \frac{\omega_c \Delta t}{2} E_x^n(i,j+1/2) \right) \end{aligned}$$

Cálculo do Campo Eléctrico (M Xu & Yuan)

calcExFieldKXY

$$
\begin{aligned}
E_x^{n+1}(i,j+1/2) &= \left(\frac{1-\frac{\Delta t\sigma_y(i,j+1/2)}{2\varepsilon_0}}{1+\frac{\Delta t\sigma_y(i,j+1/2)}{2\varepsilon_0}}\right)E_x^n(i,j+1/2) - \frac{\Delta t}{\varepsilon_0\left(1+\frac{\Delta t\sigma_y(i,j+1/2)}{2\varepsilon_0}\right)}J_x^{n+1/2}(i,j+1/2) + \\
&+ \frac{\Delta t}{\varepsilon_0\Delta y\left(1+\frac{\Delta t\sigma_y(i,j+1/2)}{2\varepsilon_0}\right)}\Big[H_{zx}^{n+1/2}(i,j+1) - H_{zx}^{n+1/2}(i,j) + \\
&+ H_{zy}^{n+1/2}(i,j+1) - H_{zy}^{n+1/2}(i,j)\Big]
\end{aligned}
$$

calcEyFieldKXY

$$
\begin{aligned}
E_y^{n+1}(i+1/2,j) &= \left(\frac{1-\frac{\Delta t\sigma_x(i+1/2,j)}{2\varepsilon_0}}{1+\frac{\Delta t\sigma_x(i+1/2,j)}{2\varepsilon_0}}\right)E_y^n(i+1/2,j) - \frac{\Delta t}{\varepsilon_0\left(1+\frac{\Delta t\sigma_x(i+1/2,j)}{2\varepsilon_0}\right)}J_y^{n+1/2}(i+1/2,j) - \\
&- \frac{\Delta t}{\varepsilon_0\Delta x\left(1+\frac{\Delta t\sigma_x(i+1/2,j)}{2\varepsilon_0}\right)}\Big[H_{zx}^{n+1/2}(i+1,j) - H_{zx}^{n+1/2}(i,j) + \\
&+ H_{zy}^{n+1/2}(i+1,j) - H_{zy}^{n+1/2}(i,j)\Big]
\end{aligned}
$$

Cálculo de H_z (Déspres & Pinto)

$$H_{zx}^{n+1/2}(i,j) = \left(\frac{1-\frac{\Delta t\sigma_x^\star(i,j)}{2\mu_0}}{1+\frac{\Delta t\sigma_x^\star(i,j)}{2\mu_0}}\right)H_{zx}^{n-1/2}(i,j) - \frac{\Delta t}{\mu_0\Delta x\left(1+\frac{\Delta t\sigma_x^\star(i,j)}{2\mu_0}\right)}\left[E_y^n(i+1/2,j) - E_y^n(i-1/2,j)\right]$$

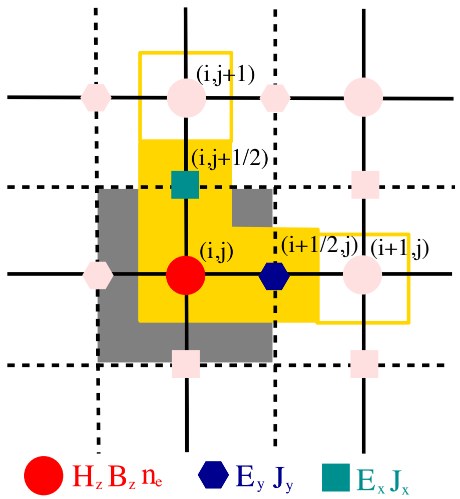
$$H_{zy}^{n+1/2}(i,j) = \left(\frac{1-\frac{\Delta t\sigma_y^\star(i,j)}{2\mu_0}}{1+\frac{\Delta t\sigma_y^\star(i,j)}{2\mu_0}}\right)H_{zy}^{n-1/2}(i,j) + \frac{\Delta t}{\mu_0\Delta y\left(1+\frac{\Delta t\sigma_y^\star(i,j)}{2\mu_0}\right)}\left[E_x^n(i,j+1/2) - E_x^n(i,j-1/2)\right]$$

Cálculo da densidade de corrente $J_{x,y}$ (Déspres & Pinto)

$$\omega_c(i,j) = \frac{\mathrm{e}}{\mathrm{m_e}} B_z(i,j) \quad \omega_p^2(i,j) = \frac{\mathrm{e}^2}{\mathrm{m_e}\varepsilon_0} n_e(i,j) \quad \alpha = 1 + \frac{\Delta t^2 \omega_p^2}{4} \quad \beta = 1 + \frac{\Delta t \omega_c}{2} \quad \gamma = \beta/\alpha \quad \delta = \frac{\Delta t \varepsilon_0 \omega_p^2}{2}$$

$$(MH)_x(i,j) = \left[H_{zx}^{n+1/2}(i,j+1) - H_{zx}^{n+1/2}(i,j) + H_{zy}^{n+1/2}(i,j+1) - H_{zy}^{n+1/2}(i,j)\right]/\Delta y$$

$$(MH)_y(i,j) = \left[H_{zx}^{n+1/2}(i+1,j) - H_{zx}^{n+1/2}(i,j) + H_{zy}^{n+1/2}(i+1,j) - H_{zy}^{n+1/2}(i,j)\right]/\Delta x$$



$$A_x(i,j) = E_x^n(i,j+1/2) + \frac{\Delta t}{\varepsilon_0}(MH)_x(i,j) - \frac{\Delta t}{2\varepsilon_0} J_x^n(i,j+1/2)$$

$$A_y(i,j) = E_y^n(i+1/2,j) - \frac{\Delta t}{\varepsilon_0}(MH)_y(i,j) - \frac{\Delta t}{2\varepsilon_0} J_y^n(i+1/2,j)$$

$$B_x(i,j) = J_x^n(i,j+1/2) + \delta E_x^n(i,j+1/2) - \beta J_y^n(i+1/2,j)$$

$$B_y(i,j) = J_y^n(i+1/2,j) + \delta E_y^n(i+1/2,j) + \beta J_x^n(i,j+1/2)$$

$$C_x(i,j) = B_x(i,j)/\alpha + \delta/\alpha A_x(i,j)$$

$$C_y(i,j) = B_y(i,j)/\alpha + \delta/\alpha A_y(i,j)$$

$$J_x^{n+1}(i,j+1/2) = \frac{1}{1+\gamma^2}\left[C_x(i,j) - \gamma C_y(i,j)\right]$$

$$J_y^{n+1/2}(i+1/2,j) = \frac{1}{1+\gamma^2}\left[C_y(i,j) + \gamma C_x(i,j)\right]$$

Cálculo do Campo Eléctrico (Déspres & Pinto)

$$
\begin{aligned}
E_x^{n+1}(i,j+1/2) &= \left(\frac{1-\frac{\Delta t \sigma_y(i,j+1/2)}{2\varepsilon_0}}{1+\frac{\Delta t \sigma_y(i,j+1/2)}{2\varepsilon_0}}\right) E_x^n(i,j+1/2) - \\
&- \frac{\Delta t}{\varepsilon_0\left(1+\frac{\Delta t \sigma_y(i,j+1/2)}{2\varepsilon_0}\right)} \frac{J_x^{n+1}(i,j+1/2)+J_x^n(i,j+1/2)}{2} + \\
&+ \frac{\Delta t}{\varepsilon_0 \Delta y\left(1+\frac{\Delta t \sigma_y(i,j+1/2)}{2\varepsilon_0}\right)} \Big[H_{zx}^{n+1/2}(i,j+1) - H_{zx}^{n+1/2}(i,j) + \\
&+ H_{zy}^{n+1/2}(i,j+1) - H_{zy}^{n+1/2}(i,j)\Big] \\
E_y^{n+1}(i+1/2,j) &= \left(\frac{1-\frac{\Delta t \sigma_x(i+1/2,j)}{2\varepsilon_0}}{1+\frac{\Delta t \sigma_x(i+1/2,j)}{2\varepsilon_0}}\right) E_y^n(i+1/2,j) - \\
&- \frac{\Delta t}{\varepsilon_0\left(1+\frac{\Delta t \sigma_x(i+1/2,j)}{2\varepsilon_0}\right)} \frac{J_y^{n+1}(i+1/2,j)+J_y^n(i+1/2,j)}{2} - \\
&- \frac{\Delta t}{\varepsilon_0 \Delta x\left(1+\frac{\Delta t \sigma_x(i+1/2,j)}{2\varepsilon_0}\right)} \Big[H_{zx}^{n+1/2}(i+1,j) - H_{zx}^{n+1/2}(i,j) + \\
&+ H_{zy}^{n+1/2}(i+1,j) - H_{zy}^{n+1/2}(i,j)\Big]
\end{aligned}
$$