# Full wave test of analytical theory for fixed frequency fluctuation reflectometry

M. Schubert[1], F. Clairet[1], F. da Silva[2], T. Gerbaud[3], E. Gusakov[4], S. Heuraux[5], A. Popov[4] and R. Sabot[1]

[1] *CEA, IRFM, F-13108 Saint-Paul-lez-Durance, France*
[2] *Instituto de Plasmas e Fusão Nuclear, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*
[3] *LPP, CNRS - UPMC (UMR-7648), Ecole Polytechnique, 91128 Palaiseau, France*
[4] *Ioffe Institute, Politekhnicheskaya 26, 194021 St.Petersburg, Russia*
[5] *IJL, CNRS (UMR-7198), Université Henri Poincaré, 54506 Vandoeuvre Cedex, France*

Fixed frequency reflectometry is used routinely to measure fluctuations of the plasma density on magnetic fusion devices [1, 2]. The absolute value of the mean square density fluctuation $<\delta n^2>$ can be determined from the measured variance of the reflectometry phase $<\delta\Phi^2>$. However, while calculating the absolute value, two particularities ought to be taken into account: Firstly, the efficiency of the density fluctuation $\delta n_\kappa$ in producing the phase fluctuation $\delta\Phi_\kappa$ strongly depends on the fluctuation's wavenumber $\kappa$. Secondly, fluctuations of density in a fusion plasma have a broad wavenumber spectrum, which is turbulent in general, but may also contain coherent parts due to MHD modes for example. Therefore, the transfer function $f_T$ between $<\delta\Phi^2>$ and $<\delta n^2>$ can vary considerably. The function $f_T$ is determined usually by using numerical simulations of the microwave propagation in a model plasma turbulence.

In this paper we present a validation of a full-wave simulation in 1D with analytic expressions that have been derived [3] by assuming a model turbulence with given radial wavenumber spectrum. The validation has two advantages: On one hand, it serves as a benchmark for full-wave codes. On the other hand, it allows a derivation of $f_T$ from first principles, which provides a sound physical basis for the whole measurement scheme.

In the following article we recall briefly the analytic expressions from [3] and we show in a simplified geometry numerical (full-wave) results. We then proceed to compare both and thus can verify all parametric dependencies. As a side effect we provide a criterion for the longest wavelength of the model fluctuation that has to be included in the simulation. Thereafter, we discuss a more realistic geometry with inhomogeneous fluctuation level and an arbitrary density profile. The non-local interaction of the density fluctuations with the microwave beam combined with the spatial inhomogeneity of the plasma density make it necessary to use an inversion algorithm for $<\delta n^2(x)>$ . We also comment the case when the measured $<\delta\Phi^2>$ are subject to error.

## Analytical formulae

The fundamentals of the following equations are published elsewhere [3]. We use O-mode, cut-off density $n_{\rm co} = m_{\rm e}\,\epsilon_0\,\omega_0^2\,e^{-2}$ and local refraction index $N(x) = \sqrt{1 - n_{\rm e}(x)/n_{\rm co}}$ . If a microwave with the vacuum wavenumber $k_0$ is launched into the plasma that contains a small amplitude, monochromatic plasma density fluctuation $\kappa_x$, then we obtain (1). An average linear density gradient is assumed, and $L$ being the length between the plasma edge and the cut-off. The validity range of (1) is $0 \ll \kappa_x < 2\,k_0$ which is the backscattering limit in a plasma. If $\kappa_x$ is replaced with the effective wavenumber $\kappa_{\rm eff}$ , then the equation (1) can also be used for a spectrum of density fluctuations. In the case of a Gaussian spectrum $\delta n_\kappa^2 \propto \exp[-l_c^2\kappa^2/4]$ the analytical solution according to [3] is displayed in (2).

$$<\!\delta\Phi^2\!> \;=\; \pi\, k_0^2\, \frac{L}{\kappa_x}\, \frac{<\delta n^2>}{n_{\rm co}^2} \qquad (1) \qquad\qquad \kappa_{\rm eff} = \frac{\sqrt{\pi}}{l_c}\, \left(\ln\left[\frac{8L}{\pi l_c}\right] + 0.711\right)^{-1} \qquad (2)$$

The correlation length $l_c$ should be small compared to the system size $L$. Similarly, the relative fluctuation level should be small to avoid secondary cut-offs $|\delta n_{\rm rms}/n_{\rm co}| \ll l_c/L$ . In order to minimise backscattering far from the cut-off, the Airy wavenumber should be large compared to the width of the fluctuation spectrum : $(k_0^2/L)^{1/3} \gg l_c^{-1}$ .

If the density and/or the fluctuation level profile are inhomogeneous and the spectrum is still a Gaussian, we deduce

$$<\!\delta\Phi^2\!> \;=\; \pi\, k_0^2 \int_{x_{\rm edge}}^{\bar{x_{\rm c}}(k_0)} G(x)\, \frac{l_c}{\sqrt{\pi}}\, \frac{<\delta n^2(x)>}{n_{\rm co}^2}\, dx \qquad (3)$$

where $\bar{x_{\rm c}}(k_0)$ is the position of the cut-off in the average density profile and $G$ is a weighting function. Far from the cut-off $|x - \bar{x_{\rm c}}| \geq l_c$ , $G$ can be approximated by the reciprocal of the refraction index squared. In the vicinity of the cut-off $|x - \bar{x_{\rm c}}| < l_c$ the analytical solution is only possible for a linear variation of $N^2$, i.e. a linear density profile in O-mode :

$$G_{\rm far}(x) = N^{-2}(x)\,, \qquad G_{\rm near}(x) = 2\frac{\sqrt{\pi}}{l_c} L_{N^2}\, \exp\left[-\frac{2(x-\bar{x_{\rm c}})^2}{l_c^2}\right]\, \mathrm{I}_0\left[\frac{2(x-\bar{x_{\rm c}})^2}{l_c^2}\right] .$$

Here, $\mathrm{I}_0(s)$ is the modified Bessel function solving $s^2\mathrm{I}_0'' + s\mathrm{I}_0' - s^2\mathrm{I}_0 = 0$ , and $L_{N^2}$ is the local gradient length at the cut-off. This definition of $G$ is a good approximation for arbitrary profiles, if the local gradient length $L_{N^2}$ does not change significantly within the distance $l_c$ .

## Full wave simulation

In order to simulate the reflectometry measurement, the Helmholtz equation for the electric field of the launched microwave is solved in the cold plasma approximation by using a 4th order Numerov scheme [4]. The plasma density is $n_{\rm e} = \bar{n}(x) + \delta n(x,t)$ , where the fluctuation $\delta n(x,t)$ is modelled in wavenumber space and Fourier transformed. In wavenumber space, we define the amplitude spectrum $\delta n_\kappa$ and use different sets of random phases in order to produce the $t$ individual "snapshots" $\delta n(x,t)$ . Since we do not want to introduce an additional average value, we set $\delta n_{\kappa=0} = 0$. The value of the wavenumber resolution $\Delta\kappa$ thus defines also the wavelength of the longest mode that is present in the simulation. Finally, the simulated $\delta\Phi_{\rm FW}^2$ (full-wave) is obtained from the variance of $\delta\Phi(t)$ in a significant number ($N \approx 50000$) of $t$ .

With this choice of the simulation setup, $\delta\Phi_{\rm FW}^2$ depends on $\Delta\kappa$ , since long wavelength fluctuations are more efficient in producing phase fluctuations (Fig. 1). We observe that

$\delta\Phi^2_{\mathrm{FW}}$ is within 10% of its final value ($\Delta\kappa \to 0$), if $\Delta\kappa \cdot L < 1$ . This criterion can already be demanding in terms of computing resources, given the plasma dimension is large and the vacuum wavelength is small.

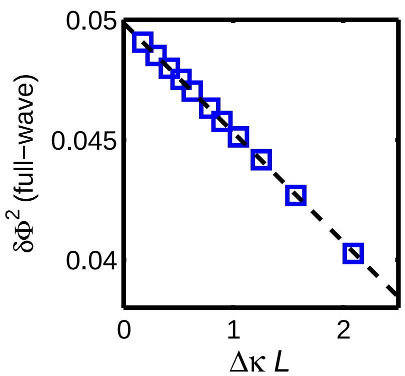



*Figure 1: Simulated phase variance as a function of $\Delta\kappa$. Note that $\Delta\kappa$ stands for both the wavenumber resolution and the wavenumber of the longest mode that is present in the simulation box. Parameters that were kept constant: $f = 47\,\mathrm{GHz}$, $L = 58\,\mathrm{cm}$, $l_c = 0.75\,\mathrm{cm}$, Gaussian spectrum, homogeneous fluctuation level $\delta n_{\mathrm{rms}}/n_{co}$=0.001.*

Hereupon, we compare $<\delta\Phi^2>$ from equations (1) and (2) with the corresponding simulation result in a linear density gradient and homogeneous turbulence. We verify the parametric dependency of $<\delta\Phi^2>$ on $k_0$, $L$, $l_c$ and $\delta n_{\mathrm{rms}} = \sqrt{<\delta n^2>}$. Figure 2 illustrates that the results correlate properly. Systematic deviations due to non-linear processes are observed for relative fluctuation levels $> 3\%$.

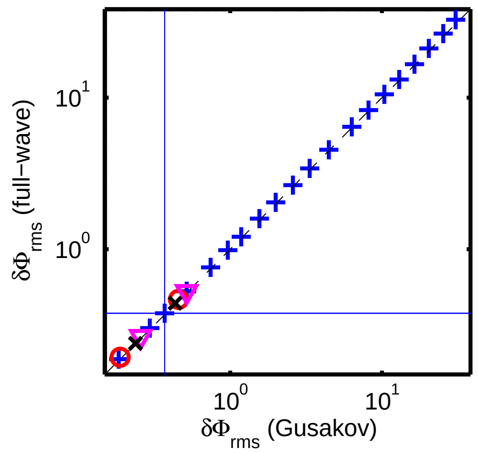
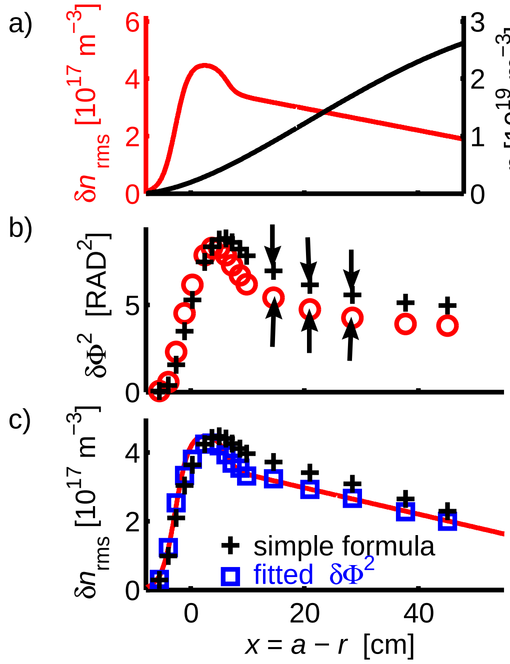



*Figure 2: Full scale comparison of the analytical formula with full-wave simulation results. Cross hair: $f = c_0 k_0/2\pi = 47\,\mathrm{GHz}$, $L$ = 58 cm, $l_c$ = 3 cm, $\delta n_{\mathrm{rms}}/n_{co}$ = 0.001 . Scan of single parameters, keeping the others constant: + (blue) $\delta n_{\mathrm{rms}}/n_{co} = 0.0005\ \ldots\ 0.1$ , $\bigtriangledown$ (magenta) $L = 32\,/\,100\,\mathrm{cm}$ , $\circ$ (red) $f = 24/58\,\mathrm{GHz}$, $\times$ (black) $l_c = 1.0/4.5\,\mathrm{cm}$.*

The most general case that we would like to discuss comprises arbitrary profiles and the corresponding formula (3). We also use a Gaussian spectrum and multiply the homogeneous data $\delta n(x,t)$ with the pre-defined envelope function $\delta n_{\mathrm{rms}}(x)$. Since the scale lengths of $\delta n_{\mathrm{rms}}(x)$ are usually large compared to $l_c$ , we neglect the effect on the spectrum.

In order to reconstruct $<\delta n^2(x)>$ from a set of $\delta\Phi^2_{\mathrm{FW}}$ (different $k_0$) we choose a least-square fit procedure. The initial guess of $<\delta n^2(x)>$ is obtained from equation (1). In this formula, for a given $k_0$ we replace $L$ with the local gradient length at the actual cut-off position. Afterwards, we use equation (3) to recalculate $<\delta\Phi^2>$ and make the difference to $\delta\Phi^2_{\mathrm{FW}}$ subject to minimisation (Fig. 3a-c). The free parameters in this process are the $<\delta n^2(x)>$ on the grid defined by the cut-off positions. Using linear interpolation in between the grid points, we obtain a very satisfying result (Fig. 3c).

It is straightforward to study the effect of error in $\delta\Phi^2_{\mathrm{FW}}$. Varying one element of the 'measured' set $\delta\Phi^2_{\mathrm{FW}}$ (Fig. 3d), we show the effect on the $<\delta n^2(x)>$ profile after the fit (Fig. 3e). If several $\delta\Phi^2_{\mathrm{FW}}$ are in error, then their contributions must be added following the laws of error propagation. Figure 3e illustrates also the limited spatial range of $\delta n_{\mathrm{rms}}(x)$ which is affected by the error.



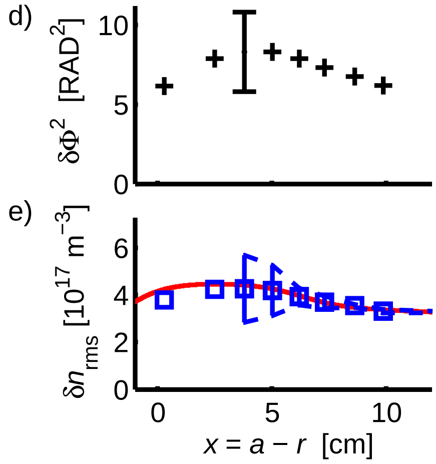




*Figure 3: a) Input density and fluctuation level. b) Open circles: $\delta\Phi^2_{\rm FW}$, 17 different frequencies, function of the corresponding cut-off location. + signs: $<\delta\Phi^2>$ reconstructed from the initial guess of $\delta n_{\rm rms}(x)$. Arrows indicate the difference that is subject to minimisation. c) + signs: initial guess of $\delta n_{\rm rms}(x)$ (see the text body). Blue squares: $\delta n_{\rm rms}(x)$ from least-squares fitting of $<\delta\Phi^2>$ data. Solid red line: input $\delta n_{\rm rms}(x)$. d) Error in the $\delta\Phi^2_{\rm FW}$ data, located at the plasma edge. e) Effect of the error on the reconstructed $\delta n_{\rm rms}(x)$, obtained numerically.*

## Conclusions

Using a 1D full-wave code we have successfully validated analytical expressions which relate the phase fluctuations in O-Mode fixed frequency reflectometry to the absolute value of the density fluctuation. If the profile of the fluctuation level is inhomogeneous, then the phase fluctuations can be described accurately by an integral expression. With this expression we demonstrated that the reconstruction of the fluctuation level profile can be obtained from a set of phase variances measured at different frequencies. Furthermore, the integral expression allowed us to study how uncertainty of the fluctuation level at the plasma edge can contribute to the error of the fluctuation level measured at the plasma center. Altogether we conclude that the measurement in the plasma center is robust. Future work will discuss the extension of this comparison between analytical formulae and numerical simulation to the X-Mode.

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